Introduction to Number Theory (M328K)

Homework # 2

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1. §1.3 # 4.

2. §1.3 # 8.

3. Use the Principle of Mathematical Induction to prove that

$$\sum_{k=1}^{n} \frac{k}{2^k} = 2 - \frac{n+2}{2^n},$$

for all $n \geq 1$.

4. A sequence x_n is defined recursively as follows: $x_1 = 100$ and

$$x_{n+1} = 11\, x_n - 10,$$

for all $n \geq 1$. Use the PMI to show that the sequence satisfies $x_n = 9 \cdot 11^n + 1$, for all $n \geq 1$.

5. A sequence L_n is defined, recursively, as follows: $L_1 = 0$, $L_2 = 4$, $L_3 = 24$ and

$$L_{n+1} = -3L_{n-2} + L_{n-1} + 3L_n,$$

for all $n \geq 3$. For each $n \in \mathbb{N}$, let G_n be $G_n = 3^n - (-1)^n - 4$. Use strong induction to show that these sequences satisfy: $L_n = G_n$, for all $n \in \mathbb{N}$.

6. §1.3 # 14.
Hint: Let $P(n)$ be the following predicate:

$$P(n) \Leftrightarrow \big(\exists x, y \in \mathbb{Z}\big)\big(x \geq 0 \wedge y \geq 0 \wedge n = 7x + 10y\big).$$

We can use strong induction to prove that $P(n)$ is true, for all $n \geq 54$. Show the following.

a) $P(54), P(55), \ldots, P(60)$ are true. Prove each statement separately.

b) $\big(\forall n \geq 60\big)\big(P(54) \wedge \cdots \wedge P(n) \rightarrow P(n+1)\big)$.

7. §1.3 # 31. (*Hint:* Use strong induction.)