Math 328K. Fall 2025
Guide for Midterm Exam 1

Prof. Hector E. Lomeli

You should know how to solve the problems in homework assignments 1–4. Focus on the following.

- **§1.1** 12, 13, 14, 18.
- **§1.2** 5, 20, 22–24.
- **§1.3** 4, 14, 29, 31.
- **§1.5** 22, 24, 37, 38, 40.
- **§3.1** 6, 14.
- **§3.3** 9–13, 24, 25, 30, 32.
- **§3.4** 1–4.

Concentrate on the following concepts.

- Natural numbers.
- Integers.
- Well-ordering principle.
- Sequence.
- Geometric sequence.
- Principle of Mathematical Induction.
- Second Principle of Mathematical Induction.
- Recursive definition of sequences.
- Divisibility.
- Division algorithm. (QRT)
- Prime numbers.
- Composite numbers.
- Greatest common divisor.
- Linear combination.
- Bezout's theorem.
- Relatively primes.
- Euclidean algorithm.

Do the following practice questions.

1) Let $b_1, b_2, \ldots$ be the sequence given by $b_k = k^5 - (k-1)^5$. Let $n \in \mathbb{N}$. Show the following.

a) $10 \mid b_k - 1$, for all $k \in \mathbb{N}$. (*Hint:* use that $2 \mid k^2 - k$.)

b) There exists an integer q such that $\displaystyle\sum_{k=1}^{n} b_k = 10q + n$.

c) $10 \mid n^5 - n$. (*Hint:* Use telescoping sums to compute $\displaystyle\sum_{k=1}^{n} b_k$.)

2) A sequence is defined, recursively, as follows: $L_1 = 0$, $L_2 = 4$, $L_3 = 24$ and

$$L_{n+1} = -3L_{n-2} + L_{n-1} + 3L_n,$$

for all $n \geq 3$. For each $n \in \mathbb{N}$, let G_n be $G_n = 3^n - (-1)^n - 4$. Show that the these sequences satisfy: $L_n = G_n$, for all $n \in \mathbb{N}$.

3) Let $a, b \in \mathbb{Z}$ and $d \in \mathbb{N}$. Assume that $d \mid a - b$. Use mathematical induction to prove that $d \mid a^n - b^n$, for all $n \in \mathbb{N}$.

4) Let $m \in \mathbb{N}$. If $a \in \mathbb{Z}$, we will denote by $a \bmod m$ the remainder of dividing a by m.

Let $x, y \in \mathbb{Z}$ be two integers such that $m \mid x - y$. Prove that $x \bmod m = y \bmod m$. This is, if $m \mid x - y$ then both integers have the same remainder when divided by m.
(*Hint:* Use uniqueness in the QRT.)

5) Let $a, b \in \mathbb{N}$ be positive integers such that $(a, b) = c$. Suppose that $c \neq b$. Prove the following.

a) $0 < c < b$.

b) There exists $x \in \mathbb{Z}$ such that $(ax) \bmod b = c$. (*Hint:* use Bezout's theorem.)

6) Let p be prime and $\alpha \in \mathbb{Z}$ be an integer such that $p \nmid \alpha$. Show that there exists $\beta \in \mathbb{Z}$ such that $(\alpha\beta) \bmod p = 1$. (*Hint:* Prove that $(\alpha, p) = 1$ and use Bezout's theorem.)

7) Let $a_1, a_2 \in \mathbb{Z}$ and $b \in \mathbb{N}$. Prove that the following statements are equivalent

a) $(a_1, b) = (a_2, b) = 1$.

b) $(a_1 a_2, b) = 1$.

8) Let m_1, m_2 be two relatively prime integers. Let $x_1, x_2 \in \mathbb{Z}$ such that $(x_1, m_1) = 1$, $(x_2, m_2) = 1$. We define

$$z = x_2 m_1 + x_1 m_2 \in \mathbb{Z},$$

Show that $(z, m_1) = 1$, $(z, m_2) = 1$ and $(z, m_1 m_2) = 1$.
