Problem 3. Solution 1. Let \( \varepsilon = |y^*| - \gamma > 0 \). For this fixed number, there exists \( K \in \mathbb{N} \) such that \( |y_n - y^*| < \varepsilon \), for all \( n \geq K \). The triangle inequality implies that \( |y^*| \leq |y_n| + |y^* - y_n| \), for all \( n \). Therefore, \( \gamma = |y^*| - \varepsilon \leq |y_n| + |y^* - y_n| - \varepsilon < |y_n| \), for all \( n \geq K \).

Problem 3. Solution 2. If \( y_n \to y^* \) then \( |y_n| \to |y^*| \). Let \( \varepsilon = |y^*| - \gamma > 0 \). There exists \( K' \in \mathbb{N} \) such that \( ||y_n| - |y^*|| < \varepsilon \), for all \( n \geq K' \). This implies that

\[
\gamma = |y^*| - \varepsilon < |y_n| < |y^*| + \varepsilon,
\]

for all \( n \geq K' \).

Problem 4. Solution 1. Define \( \alpha = \lim_{n \to \infty} x_n \) and \( \beta = \lim_{n \to \infty} y_n \).

We will use a proof by way of contradiction. Suppose that \( \alpha > \beta \). Let \( \varepsilon \) be a number such that

\[
0 < \varepsilon < \frac{\alpha - \beta}{2}.
\]

The number \( \varepsilon \) satisfies the inequality \( \beta + \varepsilon < \alpha - \varepsilon \).

Using the definition of limit, we know that there exists \( N_1 \in \mathbb{N} \) such that \( |x_n - \alpha| < \varepsilon \), for all \( n \geq N_1 \). Also, there exists \( N_2 \in \mathbb{N} \), such that \( |y_n - \beta| < \varepsilon \), for all \( n \geq N_2 \).

Let \( n \geq \max\{N_0, N_1, N_2\} \). Then we have three inequalities: \( x_n \leq y_n \), \( y_n < \beta + \varepsilon \) and \( \alpha - \varepsilon < x_n \).

These inequalities would imply that

\[
x_n \leq y_n < \beta + \varepsilon < \alpha - \varepsilon < x_n,\]

a contradiction. Therefore \( \alpha \leq \beta \).

Problem 4. Solution 2. Consider the sequence \( z_n = y_n - x_n \). Then \( z_n \geq 0 \), for all \( n \geq N_0 \). If we let \( L = \lim_{n \to \infty} z_n \), then \( L = \lim_{n \to \infty} y_n - \lim_{n \to \infty} x_n \). It is enough to show that \( L \geq 0 \).

Given \( \varepsilon > 0 \) there exists \( N_1 \in \mathbb{N} \) such that \( |z_n - L| < \varepsilon \), for all \( n \geq N_1 \). This implies that, if \( n \geq \max\{N_0, N_1\} \), then \( 0 \leq z_n < L + \varepsilon \).

If \( L < 0 \), we can choose \( \varepsilon_0 = -L > 0 \). The previous argument implies that \( 0 < L + \varepsilon_0 < 0 \), that is a contradiction. We conclude \( L \geq 0 \) and therefore \( \lim_{n \to \infty} x_n \leq \lim_{n \to \infty} y_n \).
Problem 5 a) We have seen before that \( \frac{2^n}{n!} \leq \frac{4}{n} \) for all \( n \in \mathbb{N} \). This implies that \( |y_n| \leq 4/n \), for all \( n \in \mathbb{N} \).

Problem 5 b) We know that \( 4/n \to 0 \). Using the sandwich theorem, we find that \( |y_n| \to 0 \). Therefore, \( y_n \to 0 \).

Problem 8 a) If \( |z| < m \), then \( z > -m \) and \( -z > -m \). Therefore,

\[
\left(1 + \frac{z}{n}\right)^n \geq \left(1 + \frac{z}{m}\right)^m, \quad \text{and} \quad \left(1 - \frac{z}{n}\right)^n \geq \left(1 - \frac{z}{m}\right)^m,
\]

for all integers \( n \geq m \).

Also, if \( |z| < m \) and \( n \geq m \), then both terms \( \left(1 - \frac{z}{n}\right)^n \) and \( \left(1 - \frac{z}{m}\right)^m \) are positive. This implies that

\[
\frac{1}{\left(1 - \frac{z}{n}\right)^n} \leq \frac{1}{\left(1 - \frac{z}{m}\right)^m},
\]

for all \( n \geq m \).

Finally, if \( |z| < m \leq n \), then \( 0 \leq z^2/n^2 \leq 1 \) and therefore

\[
\left(1 + \frac{z}{n}\right)^n \left(1 - \frac{z}{n}\right)^n = \left(1 - \frac{z^2}{n^2}\right)^n \leq 1.
\]

From the previous inequality we conclude that

\[
\left(1 + \frac{z}{n}\right)^n \leq \frac{1}{\left(1 - \frac{z}{n}\right)^n}.
\]

Problem 8 b) We know that the convergent sequence \((x_n)\) is bounded. Therefore, there exists a positive integer \( m \) such that \( |x_n| < m \), for all \( n \). This number \( m \) will be fixed from now on.

Consider the polynomial \( p(z) = \left(1 + \frac{z}{m}\right)^m \). We notice that, by construction, \( p(-x_n) > 0 \), for \( n \geq m \).

Using the previous part of this problem, we have that

\[
p(x_n) \leq \left(1 + \frac{x_n}{n}\right)^n \leq \frac{1}{p(-x_n)},
\]

for all \( n \in \mathbb{N} \) with \( n \geq m \). Taking the limit, we find that \( p(x_n) \to 1 \) and \( 1/p(-x_n) \to 1 \).

Using the sandwich theorem, we conclude that

\[
\left(1 + \frac{x_n}{n}\right)^n \to 1.
\]