Differential Equations with Linear Algebra (M427J)

Answers to the guide for the Final

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§3.10 # 1. The general solution is $\mathbf{x}(t) = C_1\mathbf{x}^1(t) + C_2\mathbf{x}^2(t) + C_3\mathbf{x}^3(t)$, where

$$\mathbf{x}^1(t) = \begin{bmatrix} 0 \\ e^{-2t} \\ e^{-2t} \end{bmatrix}, \qquad \mathbf{x}^2(t) = \begin{bmatrix} e^{-t} \\ e^{-t} \\ 0 \end{bmatrix}, \qquad \mathbf{x}^3(t) = \begin{bmatrix} t\,e^{-t} \\ t\,e^{-t} \\ e^{-t} \end{bmatrix}.$$

§3.10 # 2. The general solution is $\mathbf{x}(t) = C_1\mathbf{x}^1(t) + C_2\mathbf{x}^2(t) + C_3\mathbf{x}^3(t)$, where

$$\mathbf{x}^1(t) = e^{2t}\begin{bmatrix} 1-t \\ 2t - \frac{1}{2}t^2 \\ \frac{1}{2}t^2 - 3t \end{bmatrix}, \qquad \mathbf{x}^2(t) = e^{2t}\begin{bmatrix} t \\ \frac{1}{2}t^2 - t + 1 \\ 2t - \frac{1}{2}t^2 \end{bmatrix}, \qquad \mathbf{x}^3(t) = e^{2t}\begin{bmatrix} t \\ \frac{1}{2}t^2 - t \\ -\frac{1}{2}t^2 + 2t + 1 \end{bmatrix}.$$

§3.10 # 4. The general solution is $\mathbf{x}(t) = C_1\mathbf{x}^1(t) + C_2\mathbf{x}^2(t) + C_3\mathbf{x}^3(t) + C_4\mathbf{x}^4(t)$, where

$$\mathbf{x}^1(t) = e^{2t}\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \qquad \mathbf{x}^2(t) = e^{2t}\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \qquad \mathbf{x}^3(t) = e^{2t}\begin{bmatrix} -t \\ t \\ 1 \\ -t \end{bmatrix}, \qquad \mathbf{x}^4(t) = e^{2t}\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

§3.10 # 6.

$$\mathbf{x}(t) = e^{2t}\begin{bmatrix} -4t^2 - 16t + 2 \\ 6t^2 + 26t + 1 \\ -2t^2 - 10t - 1 \end{bmatrix}.$$

§3.10 # 8.

$$\mathbf{x}(t) = e^{3t}\begin{bmatrix} 1 \\ 1+t \\ 1 \\ 1+2t \end{bmatrix}.$$

§5.3 # 2. We will use $\alpha^2 = 1.14$ and $L = 2$. The heat equation has the solution

$$u(x,t) = \sum_{n=1}^{\infty} C_n\, e^{-\alpha^2\lambda_n\, t} \sin\left(\frac{n\pi}{L}x\right) = \sum_{n=1}^{\infty} C_n\, e^{-\alpha^2\lambda_n\, t} \sin\left(\frac{n\pi}{2}x\right),$$

where $\lambda_n = n^2\pi^2/L^2 = n^2\pi^2/4$. At $t = 0$, we should find constants C_n such that

$$u(x,0) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi}{2}x\right) = \sin\left(\frac{\pi}{2}x\right) - 3\sin\left(2\pi\, x\right).$$

Comparing both sides, we get that $C_1 = 1$, $C_4 = -3$, and the rest are zero. We conclude that

$$u(x,t) = e^{-\alpha^2\lambda_1\, t}\sin\left(\frac{\pi}{2}x\right) - 3e^{-\alpha^2\lambda_4\, t}\sin\left(2\pi\, x\right) = e^{-0.285\pi^2\, t}\sin\left(\frac{\pi}{2}x\right) - 3e^{-4.56\pi^2\, t}\sin\left(2\pi\, x\right).$$

§5.6 # 2 (d). We will use $\alpha^2 = 1.14$ and $L = 2$. The heat equation has the solution

$$u(x,t) = \sum_{n=1}^{\infty} C_n \, e^{-\alpha^2 \lambda_n t} \sin\left(\frac{n\pi}{L} x\right) = \sum_{n=1}^{\infty} C_n \, e^{-\alpha^2 \lambda_n t} \sin\left(\frac{n\pi}{2} x\right),$$

where $\lambda_n = n^2\pi^2/L^2 = n^2\pi^2/4$. At $t = 0$, we should find constants C_n such that

$$u(x,0) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi}{2} x\right) = f(x),$$

where

$$f(x) = \begin{cases} 0, & \text{if } 0 \le x < 1, \\ 75, & \text{if } 1 \le x \le 2. \end{cases}$$

Using the formula

$$C_n = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi}{L} x\right) dx = \int_0^2 f(x) \sin\left(\frac{n\pi}{2} x\right) dx,$$

we find

$$C_n = \frac{150\left(\cos\left(\frac{\pi n}{2}\right) - \cos(\pi n)\right)}{\pi n}.$$

Practice Problems

1) Using the method of eigenvectors, we find the following fundamental matrix of solutions.

$$\Phi(t) = \begin{bmatrix} e^{2t} & 2e^{-t} & 2e^{t} \\ 0 & e^{-t} & e^{t} \\ e^{2t} & 0 & e^{t} \end{bmatrix}.$$

We get that

$$\Phi(0) = \begin{bmatrix} 1 & 2 & 2 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}, \qquad \text{and} \qquad \Phi(0)^{-1} = \begin{bmatrix} 1 & -2 & 0 \\ 1 & -1 & -1 \\ -1 & 2 & 1 \end{bmatrix}.$$

Using the formula $e^{tA} = \Phi(t)\Phi(0)^{-1}$, we conclude that

$$e^{tA} = \begin{bmatrix} e^{2t} + 2e^{-t} - 2e^{t} & -2e^{2t} - 2e^{-t} + 4e^{t} & -2e^{-t} + 2e^{t} \\ e^{-t} - e^{t} & -e^{-t} + 2e^{t} & -e^{-t} + e^{t} \\ e^{2t} - e^{t} & -2e^{2t} + 2e^{t} & e^{t} \end{bmatrix}.$$

2) $\mathbf{x}(t) = \begin{bmatrix} 8t + 6e^{-t} - 4 \\ 8t \\ 4 - 3e^{-t} \end{bmatrix}.$

3) $e^{tB} = \begin{bmatrix} e^{t} & 0 & 0 \\ -4e^{t}t - e^{-t} + e^{t} & \frac{1}{2}e^{-t} + \frac{1}{2}e^{t} & \frac{1}{2}e^{t} - \frac{1}{2}e^{-t} \\ -4e^{t}t + e^{-t} - e^{t} & \frac{1}{2}e^{t} - \frac{1}{2}e^{-t} & \frac{1}{2}e^{-t} + \frac{1}{2}e^{t} \end{bmatrix}.$

4) $e^{tM} = \begin{bmatrix} e^{-t} & 0 & 0 \\ \frac{1}{2}(t^2 - 6t)e^{-t} & e^{-t} & t\,e^{-t} \\ t\,e^{-t} & 0 & e^{-t} \end{bmatrix}.$

5) We will use $\alpha^2 = 1$ and $L = 2$. The heat equation has the solution

$$u(x,t) = \sum_{n=1}^{\infty} C_n e^{-\alpha^2 \lambda_n t} \sin\left(\frac{n\pi}{L}x\right) = \sum_{n=1}^{\infty} C_n e^{-\lambda_n t} \sin\left(\frac{n\pi}{2}x\right),$$

where

$$\lambda_n = \frac{n^2\pi^2}{L^2} = \frac{n^2\pi^2}{4},$$

and

$$C_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi}{L}x\right)\,dx = \int_0^2 f(x)\sin\left(\frac{n\pi}{2}x\right)\,dx.$$

To compute the general coefficient, we can split the integral

$$C_n = \int_0^1 f(x)\sin\left(\frac{n\pi}{2}x\right)\,dx + \int_1^2 f(x)\sin\left(\frac{n\pi}{2}x\right)\,dx$$

$$= \int_0^1 x\sin\left(\frac{n\pi}{2}x\right)\,dx + \int_1^2 (2-x)\sin\left(\frac{n\pi}{2}x\right)\,dx = \frac{8}{\pi^2 n^2}\sin\left(\frac{\pi n}{2}\right).$$

The first coefficients are:

$$C_1 = \frac{8}{\pi^2}, \quad C_2 = 0, \quad C_3 = -\frac{8}{9\pi^2}, \quad C_4 = 0,$$

$$C_5 = \frac{8}{25\pi^2}, \quad C_6 = 0, \quad C_7 = -\frac{8}{49\pi^2}, \quad C_8 = 0, \quad C_9 = \frac{8}{81\pi^2}, \quad C_{10} = 0.$$

6) The characteristic polynomial of A is $P_A(\lambda) = \lambda^2 - 5\lambda + 6$. The answer is

$$\mathbf{x}(t) = \begin{bmatrix} be^{2t} - 5e^{3t} \\ be^{2t} - e^{2t} - 5e^{3t} \end{bmatrix}.$$