Differential Equations with Linear Algebra (M427J)

Answers to guide 2

Fall 2025

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Practice Problems

1) In all these examples, the part that has constants will give us a clue about the linear operator and particular solution. We need to guess the roots and the characteristic polynomial.

a) **Roots:** $r = 0$ and $r = 1$. **Characteristic equation:** $r(r-1) = r^2 - r = 0$. **Linear operator:** $L[y] = y'' - y'$. **Particular solution:** $\Psi(t) = -\frac{1}{2}t^2 + 4t$.
Plugging the particular solution in the operator, we find $L[\Psi(t)] = t - 5$. The equation is

$$y'' - y' = t - 5.$$

b) **Roots:** $r = 1$ and $r = 2$. **Characteristic equation:** $(r-1)(r-2) = r^2 - 3r + 2 = 0$. **Linear operator:** $L[y] = y'' - 3y' + 2y$. **Particular solution:** $\Psi(t) = 3t + 1$.
Plugging the particular solution in the operator, we find $L[\Psi(t)] = 6t - 7$. The equation is

$$y'' - 3y' + 2y = 6t - 7.$$

c) **Roots:** $r = -2+i$ and $r = -2-i$. **Characteristic equation:** $(r+2-i)(r+2+i) = r^2+4r+5 = 0$. **Linear operator:** $L[y] = y'' + 4y' + 5y$. **Particular solution:** $\Psi(t) = 3t^3 - 2t^2 + t$.
Plugging the particular solution in the operator, we find $L[\Psi(t)] = 15t^3 + 26t^2 + 7t$. The equation is

$$y'' + 4y' + 5y = 15t^3 + 26t^2 + 7t.$$

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2) **a)** $y(t) = C_1 \cos(2t) + C_2 \sin(2t) - \frac{1}{4} t \cos(2t)$.

b) $y(t) = C_1 e^{-t} + C_2 e^{3t} - 3t^2 + 4t - \frac{14}{3}$.

c) $y(t) = C_1 e^{-t} + C_2 e^{2t} - \frac{4}{3} t e^{-t}$.

d) $y(t) = C_1 e^{-t} + C_2 e^{3t} - \frac{1}{2} e^{t} + 2\sin(t) - \cos(t)$.

e) $y(t) = C_1 e^{-3t} \cos(2t) + C_2 e^{-3t} \sin(2t) + e^{-2t} \cos(2t) + 4e^{-2t} \sin(2t)$.

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3) **a)** $y(t) = C_1 e^{-t} + C_2 e^{t} + C_3 e^{2t}$.

b) $y(t) = C_1 e^{t} + C_2 e^{4t} \cos(t) + C_3 e^{4t} \sin(t)$.

c) $y(t) = C_1 \cos(2t) + C_2 \sin(2t) + C_3 t \cos(2t) + C_4 t \sin(2t)$.

d) $y(t) = C_1 e^{-t} + C_2 t e^{-t} + C_3 e^{t} + C_4 t e^{t}$.

e) $y(t) = C_1 e^{-t} \cos(2t) + C_2 e^{-t} \sin(2t) + C_3 e^{t} + C_4 + C_5 t$.

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4) **a)** $C_1\, e^{-2t} + C_2\, e^{-2t}\cos(t) + C_3\, e^{-2t}\sin(t)$.

b) $v(t) = e^{-2t}(1 - \cos(t))$.

c) $\Psi(t) = e^{-2t}\left(\frac{t^2}{2} + \cos(t) - 1\right)$.

d) $y(t) = C_1\, e^{-2t} + C_2\, e^{-2t}\cos(t) + C_3\, e^{-2t}\sin(t) + \frac{1}{2}t^2 e^{-2t}$.

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5) $A = \begin{bmatrix} 0 & 2 & 4 \\ 0 & 1 & -2 \\ 5 & 0 & 1 \end{bmatrix}$.

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6) Only two solutions are possible: $\alpha = 6$ and $\alpha = -6$.

- For $\alpha = 6$, we can use $C_1 = 3$ and $C_2 = 2$. In this case, we have

$$C_1\,\mathbf{v}_1 + C_2\,\mathbf{v}_2 = C_1\begin{bmatrix}\alpha\\-4\end{bmatrix} + C_2\begin{bmatrix}-9\\\alpha\end{bmatrix} = 3\begin{bmatrix}6\\-4\end{bmatrix} + 2\begin{bmatrix}-9\\6\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix} = \mathbf{0}.$$

- For $\alpha = -6$, we can use $C_1 = -3$ and $C_2 = 2$. In this case, we have

$$C_1\,\mathbf{v}_1 + C_2\,\mathbf{v}_2 = C_1\begin{bmatrix}\alpha\\-4\end{bmatrix} + C_2\begin{bmatrix}-9\\\alpha\end{bmatrix} = -3\begin{bmatrix}-6\\-4\end{bmatrix} + 2\begin{bmatrix}-9\\-6\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix} = \mathbf{0}.$$

In both cases, the answer is not unique.

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8) One possible solution is $C_1 = -4$, $C_2 = 5$, $C_3 = 1$.