Differential Equations with Linear Algebra (M427J)

Answers to guide 3

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1) We can do a second row expansion and get

$$(-x)\begin{vmatrix} -2 & 3 & 5 \\ 2 & -8 & 0 \\ 0 & -5 & 2 \end{vmatrix} + y\begin{vmatrix} 1 & 3 & 5 \\ 1 & -8 & 0 \\ 2 & -5 & 2 \end{vmatrix} + (-z)\begin{vmatrix} 1 & -2 & 5 \\ 1 & 2 & 0 \\ 2 & 0 & 2 \end{vmatrix} + w\begin{vmatrix} 1 & -2 & 3 \\ 1 & 2 & -8 \\ 2 & 0 & -5 \end{vmatrix} = 30x + 33y + 12z.$$

2) $\mathbf{x}(\pi/8) = \begin{bmatrix} -3e^{-\pi/8} \\ 2e^{-\pi/8} \end{bmatrix}$.

3) Using the method of eigenvectors, we find three linearly independent solutions

$$\mathbf{x}^1(t) = \begin{bmatrix} e^{-2t} \\ 2e^{-2t} \\ -e^{-2t} \end{bmatrix}, \qquad \mathbf{x}^2(t) = \begin{bmatrix} \cos(2t) \\ \sin(2t) \\ \cos(2t) \end{bmatrix}, \qquad \mathbf{x}^3(t) = \begin{bmatrix} \sin(2t) \\ -\cos(2t) \\ \sin(2t) \end{bmatrix}.$$

The general solution is $\mathbf{x}(t) = C_1\mathbf{x}^1(t) + C_2\mathbf{x}^2(t) + C_3\mathbf{x}^3(t)$. At $t = 0$, we get the equation.

$$\mathbf{x}(0) = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = C_1\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} + C_2\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + C_3\begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}.$$

Solving, we conclude that $C_1 = 0, C_2 = 1, C_3 = 0$. The solution is

$$\mathbf{x}(t) = \begin{bmatrix} \cos(2t) \\ \sin(2t) \\ \cos(2t) \end{bmatrix}.$$

4) The matrix A has four eigenvalues: $1, 2$, and $\pm i$. Using the method of eigenvectors, we find that

$$\mathbf{x}(t) = \begin{bmatrix} \cos(t) \\ e^t + 2\sin(t) \\ 3e^t - e^{2t} + \cos(t) \\ -\sin(t) \end{bmatrix}.$$

5) The exponential is

$$e^{tA} = \frac{1}{4}\begin{bmatrix} 3 + e^{-4t} & 1 - e^{-4t} \\ 3 - 3e^{-4t} & 1 + 3e^{-4t} \end{bmatrix}.$$

6) Using the method of eigenvectors, we find the following fundamental matrix of solutions.

$$\Phi(t) = \begin{bmatrix} e^{2t} & 2e^{-t} & 2e^{t} \\ 0 & e^{-t} & e^{t} \\ e^{2t} & 0 & e^{t} \end{bmatrix}.$$

We get that

$$\Phi(0) = \begin{bmatrix} 1 & 2 & 2 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}, \qquad \text{and} \qquad \Phi(0)^{-1} = \begin{bmatrix} 1 & -2 & 0 \\ 1 & -1 & -1 \\ -1 & 2 & 1 \end{bmatrix}.$$

Using the formula $e^{tA} = \Phi(t)\Phi(0)^{-1}$, we conclude that

$$e^{tA} = \begin{bmatrix} e^{2t} + 2e^{-t} - 2e^{t} & -2e^{2t} - 2e^{-t} + 4e^{t} & -2e^{-t} + 2e^{t} \\ e^{-t} - e^{t} & -e^{-t} + 2e^{t} & -e^{-t} + e^{t} \\ e^{2t} - e^{t} & -2e^{2t} + 2e^{t} & e^{t} \end{bmatrix}.$$

7) $\mathbf{x}(t) = \begin{bmatrix} 4e^{-t} - 5e^{5t} \\ 2e^{-t} \\ -5e^{5t} \end{bmatrix}.$