Math 427J. Fall 2025

Final Cheat Sheet

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1. Linear equations

If we have the equation

$$\frac{dy}{dt} + a(t)y = b(t),$$

we can use the integrating factor

$$\mu(t) = \exp\left(\int a(t)\,dt\right),$$

to get the equation

$$\frac{d}{dt}\left(\mu(t)y(t)\right) = \mu(t)b(t).$$

2. Separable equations

If we have the equation $\frac{dy}{dt} = \frac{g(t)}{f(y)}$, then $y(t)$ satisfies

$$\int f(y)\,dy = \int g(t)\,dt + C$$

where C is a constant.

3. Exact equations

If we have the equation

$$M(t,y) + N(t,y)\frac{dy}{dt} = 0,$$

and $\partial M/\partial y = \partial N/\partial t$, then we can find ϕ such that

$$\frac{\partial \phi}{\partial t} = M, \qquad \frac{\partial \phi}{\partial y} = N.$$

In that case, y satisfies $\phi(t, y(t)) = C$, where C is a constant.

4. The Wronskian

The Wronskian of two functions y_1 and y_2 is the function

$$W(t) = W[y_1, y_2](t) = \begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix}.$$

If y_1 and y_2 are two solutions of $y'' + p(t)y' + q(t)y = 0$, then the Wronskian $W(t) = W[y_1, y_2](t)$ satisfies the linear equation

$$W'(t) + p(t)W(t) = 0.$$

The Wronskian of two solutions of a second-order equation is always zero or never zero.

5. Linear equation with constant coefficients

If $L[y] = a\,y'' + b\,y' + c\,y = 0$ has constant coefficients, the characteristic equation is $ar^2 + br + c = 0$. The discriminant of the equation is $D = b^2 - 4ac$.

6. Two real roots case

If $D > 0$ the characteristic equation has two real roots r_1 and r_2. The differential equation has a fundamental set of solutions of the form $\{y_1, y_2\}$, where

$$y_1(t) = e^{r_1 t}, \qquad y_2(t) = e^{r_2 t}.$$

The wronskian is $W(t) = (r_2 - r_1)e^{(r_1+r_2)t}$.

7. Complex roots

If $D < 0$ the characteristic equation has two complex roots $\alpha + i\beta$ and $\alpha - i\beta$. The differential equation has a fundamental set of solutions of the form $\{y_1, y_2\}$, where

$$y_1(t) = e^{\alpha t}\cos\beta t, \qquad y_2(t) = e^{\alpha t}\sin\beta t.$$

The wronskian is $W(t) = \beta e^{2\alpha t}$.

8. Repeated root case

If $D = 0$ the characteristic equation has one repeated root equal to $r = -b/2a$. The differential equation has a fundamental set of solutions of the form $\{y_1, y_2\}$, where

$$y_1(t) = e^{rt} \qquad y_2(t) = t\,e^{rt}.$$

The wronskianis $W(t) = e^{2rt}$.

9. Variation of parameters

Given the non-homogeneous equation $L[y] = g(t)$, we can find a particular solution $\Psi(t)$ of the form

$$\Psi(t) = u_1(t)y_1(t) + u_2(t)y_2(t),$$

where $\{y_1(t), y_2(t)\}$ is a fundamental set of solutions of the homogeneous equation $L[y] = 0$. The functions $u_1(t), u_2(t)$ satisfy the equations

$$u_1'(t) = \frac{-y_2(t)g(t)}{W[y_1, y_2](t)},$$

$$u_2'(t) = \frac{y_1(t)g(t)}{W[y_1, y_2](t)}.$$

10. Guessing method

Given the non-homogeneous equation $L[y] = g(t)$, we can guess the form of a particular solution $\Psi(t)$, in the following cases.

1. If $g(t) = P(t)$, where $P(t)$ is a polynomial of degree n, then

   $$\Psi(t) = t^s\big[A_0 + A_1 t + \cdots + A_n t^n\big],$$

   where s is the number of times $r = 0$ is a root of the characteristic equation.

2. If $g(t) = P(t)\,e^{\alpha t}$, where $P(t)$ is a polynomial of degree n, then

   $$\Psi(t) = t^s\big[A_0 + A_1 t + \cdots + A_n t^n\big]\,e^{\alpha t},$$

   where s is the number of times $r = \alpha$ is a root of the characteristic equation.

3. If $P(t)$ is a polynomial of degree n, and $g(t) = P(t)\,e^{\alpha t}\cos(\beta t)$ or $g(t) = P(t)\,e^{\alpha t}\sin(\beta t)$, then $\Psi(t) =$

   $$t^s\Big[\big(A_0 + A_1 t + \cdots + A_n t^n\big)\,e^{\alpha t}\cos(\beta t) + \big(B_0 + B_1 t + \cdots + B_n t^n\big)\,e^{\alpha t}\sin(\beta t)\Big]$$

   where s is the number of times $r = \alpha + i\,\beta$ is a root of the characteristic equation.

11. Linear system

A linear system is an equation of the form $\dot{\mathbf{x}} = A\mathbf{x}$, where A is an $n \times n$ matrix.

12. Eigenvectors

An eigenvector of an $n \times n$ matrix A is a vector $\mathbf{v}$ such that $\mathbf{v} \neq \mathbf{0}$ and $A\mathbf{v} = \lambda\mathbf{v}$, where λ is a scalar that we will call an eigenvalue. If λ is an eigenvalue of A, then any non-trivial solution $\mathbf{v}$ of $(A - \lambda I)\mathbf{v} = \mathbf{0}$ is an eigenvector.

13. Characteristic polynomial

- The set of all eigenvalues is equal to the roots of the following degree-n polynomial: $p_A(\lambda) = \det(A - \lambda I)$.
- If A is 2×2, then $p_A(\lambda) = \lambda^2 - \operatorname{tr}(A)\lambda + \det(A)$.
- The coefficient of λ^n is $(-1)^n$. The constant coefficient is $\det(A)$. This is, $p_A(\lambda) = (-1)^n \lambda^n + \cdots + \det(A)$.

14. Real eigenvalues, all different

If $p_A(\lambda)$ has n different real roots $\lambda_1, \ldots, \lambda_n$, then A has n linearly independent eigenvectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$. In that case, the general solution of $\dot{\mathbf{x}} = A\mathbf{x}$ is

$$\mathbf{x}(t) = C_1\mathbf{x}^1(t) + \cdots + C_n\mathbf{x}^n(t),$$

where the solutions $\{\mathbf{x}^1(t), \ldots, \mathbf{x}^n(t)\}$ are a basis of the form $\mathbf{x}^k(t) = e^{\lambda_k t}\mathbf{v}_k$.

15. Complex roots

If $p_A(\lambda)$ has a complex root of the form $\lambda = \alpha + i\beta$, then A has an eigenvector of the form $\mathbf{v} = \mathbf{r} + i\mathbf{s}$, where $\mathbf{r} \neq \mathbf{0}$, and $\mathbf{s} \neq \mathbf{0}$. In addition, this generates the following two linearly independent solutions of the linear problem $\dot{\mathbf{x}} = A\mathbf{x}$.

$$\begin{aligned} \mathbf{x}^1(t) &= e^{\alpha t}\big(\cos(\beta t)\mathbf{r} - \sin(\beta t)\mathbf{s}\big), \\ \mathbf{x}^2(t) &= e^{\alpha t}\big(\sin(\beta t)\mathbf{r} + \cos(\beta t)\mathbf{s}\big). \end{aligned}$$

16. 2×2 case

If A is 2×2, we have formulas for the solution $\mathbf{x}(t)$ of the initial value problem $\dot{\mathbf{x}} = A\mathbf{x}$, with $\mathbf{x}(0) = \mathbf{x}^0$.

- If A has two real eigenvalues λ_1, λ_2, then

$$\mathbf{x}(t) = \tfrac{1}{\lambda_1 - \lambda_2}\Big(e^{\lambda_1 t}M_2 - e^{\lambda_2 t}M_1\Big)\mathbf{x}^0,$$

  where $M_1 = A - \lambda_1 I$, and $M_2 = A - \lambda_2 I$.

- If A has a complex eigenvalue $\alpha + i\beta$, then

$$\mathbf{x}(t) = e^{\alpha t}\left(\cos(\beta t)I + \frac{\sin(\beta t)}{\beta}(A - \alpha I)\right)\mathbf{x}^0.$$

- If A has a repeated eigenvalue λ, then

$$\mathbf{x}(t) = e^{\lambda t}\Big(I + t(A - \lambda I)\Big)\mathbf{x}^0.$$

17. Exponential matrix

If A is a square matrix, we define the exponential e^{tA} of the matrix tA in the following way:

$$e^{tA} = I + tA + \frac{t^2}{2!}A^2 + \frac{t^3}{3!}A^3 + \frac{t^4}{4!}A^4 + \cdots = \sum_{k=0}^{\infty} \frac{t^k}{k!}A^k.$$

where I is the identity matrix of dimension n, and A^k is the product of A with itself $k-$times. The exponential satisfies

$$\frac{d}{dt}e^{tA} = A\,e^{tA} = e^{tA}\,A.$$

18. Initial Value Problem

The function $\mathbf{x}(t) = e^{tA}\mathbf{x}^0$ solves $\dot{\mathbf{x}} = A\mathbf{x},\ \mathbf{x}(0) = \mathbf{x}^0$.

19. Exponential matrix formula

Let A be $n \times n$. If $\Phi(t)$ is a fundamental matrix of solutions for the system $\dot{\mathbf{x}} = A\mathbf{x}$, then the exponential satisfies

$$e^{tA} = \Phi(t)\Phi(0)^{-1}.$$

20. Generalized eigenvectors

If an eigenvalue λ of a square matrix A has multiplicity m, then any vector $\mathbf{v} \neq \mathbf{0}$ that satisfies $(A - \lambda I)^m\mathbf{v} = \mathbf{0}$ is a generalized eigenvector. In that case, there are m linearly independent generalized eigenvectors.

21. Repeated roots

If an eigenvalue λ of a square matrix A has multiplicity m and $\mathbf{v}_1, \ldots, \mathbf{v}_m$ are linearly independent generalized eigenvectors then, for each $\mathbf{v}_i$, the following is a solution to the linear system $\dot{\mathbf{x}} = A\mathbf{x}$.

$$\mathbf{x}^i(t) = e^{\lambda t}\left(I + t(A - \lambda I) + \cdots + \frac{t^{m-1}}{(m-1)!}(A - \lambda I)^{m-1}\right)\mathbf{v}_i.$$

Using the formula above, we can construct m different solutions $\mathbf{x}^1(t), \ldots, \mathbf{x}^m(t)$ that are linearly independent.

22. Heat equation

$$\frac{\partial u}{\partial t} = \alpha^2\frac{\partial^2 u}{\partial x^2}, \quad u(0,t) = u(L,t) = 0, \quad u(x,0) = f(x).$$

23. Solution to the Heat equation

The heat equation has the solution

$$u(x,t) = \sum_{n=1}^{\infty} C_n\, e^{-\alpha^2\lambda_n t}\sin\left(\frac{n\pi}{L}x\right),$$

where $\lambda_n = n^2\pi^2/L^2$. At $t = 0$,

$$u(x,0) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi}{L}x\right) = f(x),$$

where $C_n = \dfrac{2}{L}\displaystyle\int_0^L f(x)\sin\left(\frac{n\pi}{L}x\right)\,dx$.