Math M427J. Fall 2025
Guide for Midterm Exam 2

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Solve the following problems.

- **2.5** 2–4, 7, 13, 14.
- **2.15** 2, 4, 6, 8, 10, 12.
- **3.1** 4, 6, 8, 14, 15.
- **3.3** 1–5.
- **Extra HW6** 2, 3, 4, 5, 6.

Concentrate on the following concepts.

- Method of guessing a particular solution.
- Higher-order equations.
- Higher-order characteristic equation.
- Higher-order fundamental solutions.
- Particular solution for higher-order equations.
- Matrix. Dimension of a matrix.
- $\mathbb{R}^n$. Sum and scalar product.
- Product of a matrix and a vector.
- Product of two matrices.
- Linear combination of vectors.
- Systems of differential equations.
- Reduction of an equation of order n.
- Linear system.
- Vector space.
- Linearly independent sets.

Do the following practice questions.

1) Find second order nonhomogeneous equations for which the following are general solutions.

a) $y(t) = C_1 + C_2 e^t - \frac{1}{2}t^2 + 4t$.

b) $y(t) = C_1 e^{2t} + C_2 e^t + 3t + 1$.

c) $y(t) = e^{-2t}(C_1 \cos t + C_2 \sin t) + 3t^3 - 2t^2 + t$.

2) Find a general solution for the following equations. Use the guessing method to find particular solutions.

a) $y'' + 4y = \sin 2t$.

b) $y'' - 2y' - 3y = 9t^2$.

c) $y'' - y' - 2y = 4e^{-t}$.

d) $y'' - 2y' - 3y = 2e^t - 10 \sin t$.

e) $y'' + 6y' + 13y = 17\, e^{-2t} \cos 2t$.

f) $y'' + 9y = 6t \sin(3t) - \cos(3t)$.

3) Find a general solution for the following equations.

a) $y^{(3)} - 2y'' - y' + 2y = 0$.

b) $y^{(3)} - 9y'' + 25y' - 17y = 0$.

c) $y^{(4)} + 8y'' + 16y = 0$.

d) $y^{(4)} - 2y'' + y = 0$.

e) $y^{(5)} + y^{(4)} + 3y^{(3)} - 5y'' = 0$.

4) Consider the linear operator

$$L[y] = y^{(3)} + 6y'' + 13y' + 10y.$$

a) Find the general solution of $L[y] = 0$.

b) Solve the initial value problem

$$L[v] = 0,\ v(0) = 0, v'(0) = 0, v''(0) = 1.$$

c) Let $g(t) = t\, e^{-2t}$. Use one of the formulas

$$\Psi(t) = \int_0^t v(t-s) g(s)\, ds = \int_0^t v(s) g(t-s)\, ds,$$

to find a solution of $L[\Psi] = g(t)$.

d) Find the general solution of $L[y] = g(t)$.

5) Consider the vectors

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ -3 \\ 1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 0 \\ 3 \\ -1 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 0 \\ -5 \\ 2 \end{bmatrix},$$

and

$$\mathbf{w}_1 = \begin{bmatrix} -2 \\ -5 \\ 6 \end{bmatrix}, \quad \mathbf{w}_2 = \begin{bmatrix} 2 \\ 5 \\ -1 \end{bmatrix}, \quad \mathbf{w}_3 = \begin{bmatrix} -2 \\ -9 \\ 2 \end{bmatrix}.$$

Let A be a 3×3 matrix with the property that

$$A \cdot \mathbf{v}_1 = \mathbf{w}_1, \ A \cdot \mathbf{v}_2 = \mathbf{w}_2, \ A \cdot \mathbf{v}_3 = \mathbf{w}_3.$$

Find the matrix A. (*Hint:* The three columns of A are $A \cdot \mathbf{e}^1$, $A \cdot \mathbf{e}^2$, $A \cdot \mathbf{e}^3$, where

$$\mathbf{e}^1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{e}^2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{e}^3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

Write $\mathbf{e}^1$, $\mathbf{e}^2$, and $\mathbf{e}^3$ in terms of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$.)

6) Consider the vectors

$$\mathbf{v}_1 = \begin{bmatrix} \alpha \\ -4 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} -9 \\ \alpha \end{bmatrix}.$$

Find all possible values of α for which $\mathbf{v}_1, \mathbf{v}_2$ are linearly dependent. In each case, find a linear combination such that $C_1\mathbf{v}_1 + C_2\mathbf{v}_2 = \mathbf{0}$, where at least one of the constants C_1, C_2 is not equal to zero.

7) Let V be the set of solutions of the linear system $\dot{\mathbf{x}} = A\mathbf{x}$, where

$$A = \begin{bmatrix} -2 & 5 \\ 1 & 2 \end{bmatrix}.$$

Verify that the following functions are elements of V that are linearly independent.

$$\mathbf{x}^1(t) = \begin{bmatrix} 3e^{3t} - 5e^{-3t} \\ e^{-3t} + 3e^{3t} \end{bmatrix}, \ \mathbf{x}^2(t) = \begin{bmatrix} 2e^{3t} - 5e^{-3t} \\ e^{-3t} + 2e^{3t} \end{bmatrix}.$$

8) We continue with the previous example. Let

$$\mathbf{x}^3(t) = \begin{bmatrix} 5e^{-3t} + 2e^{3t} \\ 2e^{3t} - e^{-3t} \end{bmatrix}.$$

Then we can verify that $\mathbf{x}^3(t)$ is also an element of the vector space V. However, $\{\mathbf{x}^1, \mathbf{x}^2, \mathbf{x}^3\}$ is dependent. Find a linear combination such that

$$C_1\mathbf{x}^1(t) + C_2\mathbf{x}^2(t) + C_3\mathbf{x}^3(t) = \mathbf{0},$$

where at least one of the constants C_1, C_2, C_3 is not equal to zero. (*Hint:* substitute $t = 0$ to simplify the equations.)