Differential Equations with Linear Algebra (M427J)

Some answers to homework # 5

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§2.15 # 2. $y(t) = C_1\, e^{-t} + C_2\, e^{3t} + C_3\, e^{4t}$.

§2.15 # 4. $y(t) = C_1 \cos(t) + C_2 \sin(t) + C_3\, e^{t}$.

§2.15 # 6. The general solution is $y(t) = C_1\, e^{t} + C_2\, e^{-t} + C_3 \cos(t) + C_4 \sin(t)$. Using the initial values, we can find the constants. The solution is

$$y(t) = \frac{1}{2} e^{-t} + \frac{1}{2} \cos(t) + \frac{1}{2} \sin(t).$$

§2.15 # 8. If $y_1(t) = e^t \cos(t)$ is a solution, we have that $1 + i$ and $1 - i$ are roots of the characteristic equation. This implies that $(r - 1 - i)(r - 1 + i) = r^2 - 2r + 2$ is a factor of the characteristic equation.

Dividing $r^4 - 2r^3 + r^2 + 2r - 2$ by $r^2 - 2r + 2$, we find that the characteristic equation can be factored as

$$r^4 - 2r^3 + r^2 + 2r - 2 = (r^2 - 2r + 2)(r^2 - 1) = (r^2 - 2r + 2)(r + 1)(r - 1).$$

The general solution is $y(t) = C_1 e^t \cos(t) + C_2 e^t \sin(t) + C_3 e^{-t} + C_4 e^t$.

§2.15 # 9. Using the formula in page 263, we find that $\Psi(t) = \int_0^t \upsilon(t - s) \tan(s)\, ds$, where

$$\upsilon(t) = 1 - \cos(t).$$

We can simplify the computation, if we expand $\upsilon(t - s) = 1 - \cos(t)\cos(s) - \sin(t)\sin(s)$. We find that

$$\Psi(t) = \int_0^t \Big(1 - \cos(t)\cos(s) - \sin(t)\sin(s)\Big) \tan(s)\, ds$$

$$= \int_0^t \tan(s)\, ds - \cos(t) \int_0^t \sin(s)\, ds - \sin(t) \int_0^t \sin(s)\tan(s)\, ds.$$

After we compute the antiderivatives, we find

$$\Psi(t) = -\ln|\cos(t)| + 1 - \cos(t) + \sin(t) \cdot \ln|\sec(t) + \tan(t)|.$$

The terms $1 - \cos(t)$ are solution of the homogeneous equation, so $\Psi(t)$ can be simplified as

$$\Psi(t) = -\ln|\cos(t)| + \sin(t) \cdot \ln|\sec(t) + \tan(t)|.$$

§2.15 # 10. Using the formula in page 263, we find that $\Psi(t) = \int_0^t g(t - s)\upsilon(s)\, ds = \int_0^t \upsilon(t - s) g(s)\, ds$, where

$$\upsilon(s) = \frac{1}{4}\left(e^t - e^{-t} - 2\sin t\right).$$

§2.15 # 12. $\Psi(t) = \dfrac{2t^3}{3} - 4t - 2t\sin(t)$. (*Hint:* Use the guessing method.)

§2.15 # 14. Let $L[y] = y^{(4)} - y$. We will use the guessing method with two term. We will solve

$$L[\Psi_1] = t, \qquad L[\Psi_2] = \sin(t).$$

The characteristic polynomial of L has the following roots: $\pm 1, \pm i$. This implies that the functions Ψ_1 and Ψ_2 have the following forms:

$$\Psi_1(t) = A_0 + A_1 t, \qquad \Psi_2(t) = t\left(\bar{A}_0\cos(t) + \bar{B}_0\sin(t)\right).$$

Solving, we find $\Psi_1(t) = -t$ and $\Psi_2(t) = \frac{1}{4}t\cos(t)$. We get $\Psi(t) = \Psi_1(t) + \Psi_2(t) = -t + \frac{1}{4}t\cos(t)$.

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§3.1 # 2. We use $x_1(t) = y(t)$, $x_2(t) = y'(t)$, and $x_3(t) = y''(t)$. We find the system

$$\begin{aligned} \dot{x_1} &= x_2, \\ \dot{x_2} &= x_3, \\ \dot{x_3} &= e^t - \cos(x_1). \end{aligned}$$

§3.1 # 4. We get a system of four equations.

$$\begin{aligned} \dot{x_1} &= x_2, \\ \dot{x_2} &= -2x_1 - 3x_4, \\ \dot{x_3} &= x_4, \\ \dot{x_4} &= -3x_2 - 2x_3. \end{aligned}$$

§3.1 # 6.

$$\dot{\mathbf{x}} = \begin{bmatrix} 3 & -7 \\ 4 & 0 \end{bmatrix}\mathbf{x}, \qquad \mathbf{x}(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

§3.1 # 8.

$$\dot{\mathbf{x}} = \begin{bmatrix} 1 & 1 & -1 \\ 3 & -1 & 4 \\ -1 & -1 & 0 \end{bmatrix}\mathbf{x}, \qquad \mathbf{x}(0) = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}.$$