

Differential Equations with Linear Algebra (M427J)

Some answers to practice problems # 6

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§3.1 # 2. We use $x_1(t) = y(t)$, $x_2(t) = y'(t)$, and $x_3(t) = y''(t)$. We find the system

$$\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= x_3, \\ \dot{x}_3 &= e^t - \cos(x_1). \end{aligned}$$

§3.1 # 4. We get a system of four equations.

$$\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= -2x_1 - 3x_4, \\ \dot{x}_3 &= x_4, \\ \dot{x}_4 &= -3x_2 - 2x_3. \end{aligned}$$

§3.1 # 6.

$$\dot{\mathbf{x}} = \begin{bmatrix} 3 & -7 \\ 4 & 0 \end{bmatrix} \mathbf{x}, \quad \mathbf{x}(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

§3.1 # 8.

$$\dot{\mathbf{x}} = \begin{bmatrix} 1 & 1 & -1 \\ 3 & -1 & 4 \\ -1 & -1 & 0 \end{bmatrix} \mathbf{x}, \quad \mathbf{x}(0) = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}.$$

§3.1 # 14. We can write $\begin{bmatrix} 3 \\ -1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + 2 \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$. This implies that

$$\begin{aligned} A \begin{bmatrix} 3 \\ -1 \\ -1 \end{bmatrix} &= A \left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + 2 \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} \right) \\ &= A \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + 2A \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 6 \end{bmatrix} + 2 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 5 \\ 6 \\ 12 \end{bmatrix}. \end{aligned}$$

Extra # 1.

$$\text{a) } AC = \begin{bmatrix} 4 & -8 & -4 \\ 1 & -1 & 2 \\ -6 & 11 & 3 \end{bmatrix}.$$

$$\text{b) } CA = \begin{bmatrix} -5 & -22 \\ 1 & 11 \end{bmatrix}.$$

$$\text{c) } CAB = \begin{bmatrix} -27 & -66 \\ 12 & 33 \end{bmatrix}.$$

$$\text{d) } BCA = \begin{bmatrix} -5 & -22 \\ -2 & 11 \end{bmatrix}.$$

$$\text{e) } ABC = \begin{bmatrix} 0 & -4 & -12 \\ 3 & -3 & 6 \\ -3 & 8 & 9 \end{bmatrix}.$$

Extra # 2. $C_1 = 2, C_2 = 4, C_3 = -3.$

Extra # 3. Two solutions are possible: $a = 2, r = 2,$ or $a = 1, r = 3.$

Extra # 4.a) We can write $\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - 5 \begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix} + 3 \begin{bmatrix} 0 \\ 5 \\ 2 \end{bmatrix}.$

Extra # 4.b) Using the known values, we find that

$$\begin{aligned} A \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} &= A \left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - 5 \begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix} + 3 \begin{bmatrix} 0 \\ 5 \\ 2 \end{bmatrix} \right) \\ &= A \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - 5A \begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix} + 3A \begin{bmatrix} 0 \\ 5 \\ 2 \end{bmatrix} = \begin{bmatrix} 9 \\ 0 \\ 1 \end{bmatrix} - 5 \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} + 3 \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ -9 \\ -1 \end{bmatrix}. \end{aligned}$$

Extra # 5.a) We have the linear combination $\begin{bmatrix} 1 \\ 0 \end{bmatrix} = 5 \begin{bmatrix} 2 \\ 3 \end{bmatrix} - 3 \begin{bmatrix} 3 \\ 5 \end{bmatrix}.$

Extra # 5.b) We have the linear combination $\begin{bmatrix} 0 \\ 1 \end{bmatrix} = -3 \begin{bmatrix} 2 \\ 3 \end{bmatrix} + 2 \begin{bmatrix} 3 \\ 5 \end{bmatrix}.$

Extra # 5.c) We find the two columns of A using the linear combinations above. This is,

$$A \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 5A \begin{bmatrix} 2 \\ 3 \end{bmatrix} - 3A \begin{bmatrix} 3 \\ 5 \end{bmatrix} = 5 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} - 3 \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix} = \begin{bmatrix} -11 \\ 9 \\ 2 \end{bmatrix},$$

$$A \begin{bmatrix} 0 \\ 1 \end{bmatrix} = -3A \begin{bmatrix} 2 \\ 3 \end{bmatrix} + 2A \begin{bmatrix} 3 \\ 5 \end{bmatrix} = -3 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + 2 \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix} = \begin{bmatrix} 7 \\ -6 \\ -1 \end{bmatrix}.$$

We conclude that

$$A = \begin{bmatrix} -11 & 7 \\ 9 & -6 \\ 2 & -1 \end{bmatrix}.$$

Extra # 6. The key idea here is to notice that the definition of the product AB implies that

$$A \begin{bmatrix} 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 13 \\ 1 \\ -6 \end{bmatrix}, \quad \text{and} \quad A \begin{bmatrix} -5 \\ 2 \end{bmatrix} = \begin{bmatrix} -23 \\ -1 \\ 11 \end{bmatrix}.$$

Using these equations, we will find the columns $A \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $A \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

To do this, we need to write the vectors $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ as linear combinations of $\begin{bmatrix} 3 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} -5 \\ 2 \end{bmatrix}$.

In this case we get,

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} = 2 \begin{bmatrix} 3 \\ -1 \end{bmatrix} + \begin{bmatrix} -5 \\ 2 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 1 \end{bmatrix} = 5 \begin{bmatrix} 3 \\ -1 \end{bmatrix} + 3 \begin{bmatrix} -5 \\ 2 \end{bmatrix}.$$

Following the same steps of **Extra # 5**, we find that

$$A = \begin{bmatrix} 3 & -4 \\ 1 & 2 \\ -1 & 3 \end{bmatrix}.$$