Differential Equations with Linear Algebra (M427J)

Some answers to practice problems # 8

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§3.4 # 2. $\mathbf{x}^1(t) = \begin{bmatrix} e^{2t} \\ 2e^{2t} \\ 4e^{2t} \end{bmatrix}, \mathbf{x}^2(t) = \begin{bmatrix} \cos(t) \\ -\sin(t) \\ -\cos(t) \end{bmatrix}, \mathbf{x}^3(t) = \begin{bmatrix} \sin(t) \\ \cos(t) \\ -\sin(t) \end{bmatrix}.$

§3.4 # 4. $\mathbf{x}^1(t) = \begin{bmatrix} e^{t} \\ t\,e^{t} \\ \left(\frac{1}{2}t^2 + t\right)\, e^{t} \end{bmatrix}, \mathbf{x}^2(t) = \begin{bmatrix} 0 \\ e^{t} \\ t\,e^{t} \end{bmatrix}, \mathbf{x}^3(t) = \begin{bmatrix} 0 \\ 0 \\ e^{t} \end{bmatrix}.$

§3.5 # 4. $-t^6 + t^4 + t^3 - t.$

§3.5 # 6. 28.

§3.5 # 8. -13.

Extra # 1. We have a system of differential equations. In this case, the system can be written as

$$\begin{aligned} \dot{x_1} &= 2\,x_1, \\ \dot{x_2} &= -6\,x_1 - x_2. \end{aligned}$$

First, we solve the first equation for x_1 and get $x_1(t) = C_1\,e^{2t}$, where C_1 is a constant. We substitute this in the second equation and get

$$\dot{x_2}(t) = -6\,x_1(t) - x_2(t) = -6C_1\,e^{2t} - x_2(t).$$

We can write this equation as

$$\frac{dx_2}{dt} + x_2 = -6C_1\,e^{2t}.$$

Solving, we find $x_2(t) = -2C_1e^{2t} + C_2e^{-t}$, where C_1, C_2 are constants. We conclude that the general solution vector function is of the form

$$\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} C_1e^{2t} \\ C_2e^{-t} - 2C_1e^{2t} \end{bmatrix} = C_1 \begin{bmatrix} e^{2t} \\ -2e^{2t} \end{bmatrix} + C_2 \begin{bmatrix} 0 \\ e^{-t} \end{bmatrix}.$$

We can use the following two solutions as a basis for the space of solutions.

$$\mathbf{v}^1(t) = \begin{bmatrix} e^{2t} \\ -2e^{2t} \end{bmatrix}, \qquad \mathbf{v}^2(t) = \begin{bmatrix} 0 \\ e^{-t} \end{bmatrix}.$$

Extra # 2. If we let $\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}$, the system can be written as

$$\begin{aligned} \dot{x_1} &= x_2, \\ \dot{x_2} &= x_3, \\ \dot{x_3} &= 2\,x_1 + 3\,x_2. \end{aligned}$$

If we let $x_1(t) = y(t)$ then $x_2(t) = \dot{x_1}(t) = y'(t)$ and $x_3(t) = \dot{x_2}(t) = y''(t)$. We also have that

$$y'''(t) = \dot{x_3}(t) = 2\,x_1(t) + 3\,x_2(t) = 2\,y(t) + 3\,y'(t).$$

In this way, we get a third order equation

$$y'''(t) - 3\,y'(t) - 2\,y(t) = 0.$$

This equation has the general solution

$$y(t) = C_1\,e^{-t} + C_2\,te^{-t} + C_3\,e^{2t}.$$

From this,

$$\begin{aligned} \mathbf{x}(t) = \begin{bmatrix} y(t) \\ y'(t) \\ y''(t) \end{bmatrix} &= \begin{bmatrix} C_1\,e^{-t} + C_2\,te^{-t} + C_3\,e^{2t} \\ -C_1\,e^{-t} + C_2\,e^{-t} - C_2\,t\,e^{-t} + 2C_3\,e^{2t} \\ C_1\,e^{-t} - 2C_2\,e^{-t} + C_2\,te^{-t} + 4C_3\,e^{2t} \end{bmatrix} \\ &= C_1 \begin{bmatrix} e^{-t} \\ -e^{-t} \\ e^{-t} \end{bmatrix} + C_2 \begin{bmatrix} te^{-t} \\ e^{-t} - te^{-t} \\ te^{-t} - 2e^{-t} \end{bmatrix} + C_3 \begin{bmatrix} e^{2t} \\ 2e^{2t} \\ 4e^{2t} \end{bmatrix}. \end{aligned}$$

We can use the following three solutions as a basis for the space of solutions.

$$\mathbf{v}^1(t) = \begin{bmatrix} e^{-t} \\ -e^{-t} \\ e^{-t} \end{bmatrix}, \qquad \mathbf{v}^2(t) = \begin{bmatrix} te^{-t} \\ e^{-t} - te^{-t} \\ te^{-t} - 2e^{-t} \end{bmatrix}, \qquad \mathbf{v}^3(t) = \begin{bmatrix} e^{2t} \\ 2e^{2t} \\ 4e^{2t} \end{bmatrix}.$$

Extra # 3. Comparing both sides of the equation $\dot{\mathbf{x}} = A\mathbf{x}$, we find that $a = 1$ and $b = 2$.