A brief introduction to sofic entropy theory

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Review of classical entropy theory

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A typical setup consists of a homeomorphism $T : X \to X$ of a compact metrizable space $X$.

The **topological entropy** of $T$ is the infimum over $\epsilon > 0$ of the exponential growth rate of the number of partial orbits that can be distinguished at scale $\epsilon > 0$. 
A length-$n$ partial orbit is an $n$-tuple of the form $x = (x, Tx, T^2x, \ldots, T^{n-1}x)$. 
A partial orbit
Topological entropy ala Rufus Bowen

Let $\rho$ be a metric on $X$. 
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The $\rho_\infty$-distance on length-$n$ partial orbits is

$$\rho_\infty(x, y) = \max_{0 \leq i \leq n-1} \rho(T^i x, T^i y).$$
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$$\rho_\infty(x, y) = \max_{0 \leq i \leq n-1} \rho(T^i x, T^i y).$$

$$h(X, T) := \sup_{\epsilon > 0} \limsup_{n \to \infty} n^{-1} \log \text{cov}_\epsilon(\text{length-}n \text{ partial orbits, } \rho_\infty)$$

where $\text{cov}_\epsilon(\cdot, \rho_\infty)$ is the minimum cardinality of a collection of length-$n$ partial orbits that $\epsilon$-cover the space of all length-$n$ partial orbits.
Main Results

1. If $(X, T)$ embeds into $(Y, S)$ then $h(X, T) \leq h(Y, S)$.

2. In particular, entropy is a topological conjugacy invariant.

3. A system $(Y, S)$ is a factor of $(X, T)$ if there is a surjective $(T, S)$-equivariant map $\Phi : X \to Y$. In this case, $h(X, T) \geq h(Y, S)$.

4. (Topological entropy was defined earlier in a different way by Adler, Konheim and McAndrew in 1965).
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Overview

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What happens if we replace the acting group with a free group $\mathbb{F}_2 = \langle a, b \rangle$?
The rank 2 free group
Important example: full shifts

If $\Gamma$ is a countable group and $K$ a Borel space then the full $K$-shift is

$$\Gamma \curvearrowright K^\Gamma = \{ x : \Gamma \to K \}$$

$$(gx)_f = x_{g^{-1}f} \quad \forall g, f \in \Gamma, x \in K^\Gamma.$$
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The full shift $\mathbb{Z} \curvearrowright K^\mathbb{Z}$ has entropy $\log |K|$. 

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The Ornstein-Weiss Example

Theorem (Ornstein-Weiss, 1987)

If $F = \langle a, b \rangle$ is the rank 2 free group then the full 2-shift over $F$ factors onto the full 4-shift over $F$. 

Define $\Phi : (\mathbb{Z} / 2\mathbb{Z})^F \to (\mathbb{Z} / 2\mathbb{Z} \times \mathbb{Z} / 2\mathbb{Z})^F$ by $\Phi(x)(g) = (x(g) + x(ga), x(g) + x(gb))$. This is surjective, shift-equivariant, 2-1, continuous and a homomorphism of compact abelian groups!
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This is surjective, shift-equivariant, 2-1, continuous and a homomorphism of compact abelian groups!

*(Ornstein-Weiss, 1987)*: Is the full 2-shift over $F$ measurably conjugate to the full 4-shift?
The Ornstein-Weiss map
Factors between topological Bernoulli shifts

Theorem (Bartholdi, 2016)

If \( \Gamma \) is any non-amenable group then there exist integers \( 2 \leq m < n \) and a continuous, shift-equivariant surjective map

\[(\mathbb{Z}/m\mathbb{Z})^\Gamma \to (\mathbb{Z}/n\mathbb{Z})^\Gamma.\]
A new approach to entropy theory for $\mathbb{Z}$-actions

Consider softening the notion of partial orbit.
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An $(n, \delta)$-pseudo orbit is a tuple $x = (x_1, \ldots, x_n) \in X^n$ such that

$$\frac{1}{n} \sum_{i=1}^{n-1} \rho(Tx_i, x_{i+1}) < \delta.$$ 

Note: we are using an $\ell^1$ metric instead of an $\ell^\infty$ metric.
A pseudo-orbit
Entropy via pseudo-orbits

Theorem

\[ h(X, T) = \sup_{\epsilon > 0} \inf_{\delta > 0} \limsup_{n \to \infty} n^{-1} \log \text{cov}_\epsilon( (n, \delta)\text{-pseudo orbits}, \rho_\infty). \]
Theorem

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Proof sketch.

Use Markov's inequality to show that any \((n, \delta)\)-pseudo orbit is approximately shadowed by a union of a few long partial orbits.
A **periodic orbit with period** \( n \) is a tuple \((x, Tx, \ldots, T^{n-1}x)\) such that \( T^n x = x \) (up to cyclic reordering).
A periodic orbit
Entropy via periodic orbits?

The exponential rate of growth of the number of periodic points that can be distinguished at scale $\epsilon$ (and then send $\epsilon \downarrow 0$) is a lower bound for entropy.

But in general, it is not equal to entropy.
How to compute entropy

pseudo orbits

pseudo periodic orbits

partial orbits

periodic orbits
Pseudo-periodic orbits

An \((n, \delta)\)-pseudo-periodic orbit is a tuple \(x = (x_1, \ldots, x_n) \in X^n\) (up to cyclic order) such that

\[
\frac{1}{n} \sum_{i=1}^{n} \rho(Tx_i, x_{i+1}) < \delta
\]

(indices mod \(n\)).
Pseudo-periodic orbits

...... = errors
Entropy via pseudo-periodic orbits

Theorem

\[ h(X, T) = \sup_{\epsilon > 0} \inf_{\delta > 0} \limsup_{n \to \infty} n^{-1} \log \cov_{\epsilon}((n, \delta)\text{-pseudo-periodic orbits}, \rho_{\infty}) \]
Entropy via pseudo-periodic orbits

**Theorem**

\[ h(X, T) = \sup_{\epsilon > 0} \inf_{\delta > 0} \limsup_{n \to \infty} n^{-1} \log \text{cov}_{\epsilon}((n, \delta)-\text{pseudo-periodic orbits}, \rho_{\infty}) \]

Aside: pseudo-periodic orbits have also been called **microstates**, **good maps** or **sofic models for the action**.
Let $\Gamma$ be a countable group, $\Gamma \curvearrowright X$ an action by homeomorphisms.
A first step towards sofic entropy

Let $\Gamma$ be a countable group, $\Gamma \curvearrowright X$ an action by homeomorphisms.

**Preliminary definition**: an **pseudo-periodic orbit** consists of an action $\Gamma \curvearrowright \sigma V$ on a finite set and a map $\phi : V \to X$ that is approximately equivariant in an $\ell^1$-sense:

$$|V|^{-1} \sum_{v \in V} \rho \left( \phi(\sigma(g)v), g\phi(v) \right) < \delta \quad \forall g \in F$$

where $F \subset \Gamma$ is finite.

More precisely, this is a $(\sigma, \delta, F)$-**pseudo-periodic orbit**.
A pseudo-periodic orbit of a free group action
Let $\Sigma = \{ \Gamma \curvearrowright^\sigma V_n \}$ be a sequence of actions on finite sets.
A first step towards sofic entropy

Let $\Sigma = \{\Gamma \bowtie \sigma^n V_n\}$ be a sequence of actions on finite sets.

**Preliminary definition**: the **sofic entropy of** $\Gamma \bowtie X$ with respect to $\Sigma$ is

$$h_\Sigma(\Gamma \bowtie X) := \sup_{\epsilon > 0} \inf_{\delta > 0} \inf_{F \in \Gamma} \limsup_{n \to \infty} |V_n|^{-1} \log \text{cov}_\epsilon((\sigma_n, \delta, F)-\text{pseudo-periodic orbits}, \rho_{\infty}).$$
Main Results (Kerr-Li, 2011)

1. If $\Gamma \curvearrowright X$ embeds into $\Gamma \curvearrowright Y$ then $h_\Sigma(\Gamma \curvearrowright X) \leq h_\Sigma(\Gamma \curvearrowright Y)$.

2. In particular, $\Sigma$-entropy is a topological conjugacy invariant.

3. $h_\Sigma(\Gamma \curvearrowright \mathbb{Z}) = \log |\mathbb{Z}|$.

4. If $\Gamma$ is amenable then sofic entropy agrees with classical entropy.
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A boring example

Suppose $V_n$ is a single point for all $n$. 
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This isn’t what is usually meant by entropy.

To fix this, require that the actions $\Gamma \curvearrowright^{\sigma_n} V_n$ witness $\Gamma$ in the sense that:
for all $g \in \Gamma \setminus \{1\}$,

$$|V_n|^{-1} \# \{v \in V_n : \sigma_n(g)v \neq v \} \rightarrow 1 \text{ as } n \rightarrow \infty.$$

With this assumption, $\Sigma$ is said to be a **sofic approximation** to $\Gamma$. 
A curious example

Given an action \( \Gamma \) on a set \( V \), form the graph on \( V \):

\[
\Gamma = \langle a, b \rangle
\]

\[
V = \{0, 1, 2, 3\}
\]
A curious example

Let's compute the sofic entropy of $\Gamma \cong \{0,1\}$

Suppose each $\Gamma \cong \mathbb{Z}$ gives a bi-partite graph.
Then there exists a pseudo-periodic orbit (for any choice of $\delta,F$) and $h_\mathcal{F}(\Gamma \approx \{0,1\}) = 0.$
A curious example

Conversely, if there is a \((\sigma, \delta, \{a, b, a^{-1}, b^{-1}\})\)-pseudo-periodic orbit,

\[ \phi : V_n \to \{0, 1\} \]

then there is a \(\delta\)-almost bi-partition \(\{\phi^{-1}(0), \phi^{-1}(1)\}\) of the graph of \(\Gamma \curvearrowright V_n\).
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So, if the graphs of \(\Gamma \vartriangleleft V_n\) do not have almost bi-partitions then

\[
h_{\Sigma}(\mathbb{F}_2 \vartriangleleft \{0, 1\}) = -\infty.
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A curious example

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So, if the graphs of \(\Gamma \bowtie V_n\) do not have almost bi-partitions then

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Recap: Sofic entropy depends on the choice of sofic approximation and it can be \( -\infty \).
The Ornstein-Weiss example revisited

The reason entropy increases under the Ornstein-Weiss map is because it is not always possible to lift a pseudo-periodic orbit.
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$$\exists \widetilde{\phi} \rightarrow \phi \rightarrow (\mathbb{Z}/2\mathbb{Z})^{F_2}$$

$$V_n \rightarrow (\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z})^{F_2}$$
A cohomological interpretation of the Ornstein-Weiss example

Given

\[ \phi : V_n \to (\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z})^{F_2} \]

let

\[ \phi^{\text{edge}} : E_n \to (\mathbb{Z}/2\mathbb{Z}) \]

be the map defined on the *edges* of the action graph by

\[ \phi(v)_e = (\phi^{\text{edge}}(v, va), \phi^{\text{edge}}(v, vb)) \].
A cohomological interpretation of the Ornstein-Weiss example

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be the map defined on the edges of the action graph by

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A pseudo-periodic orbit \( \phi : V_n \to (\mathbb{Z} / 2\mathbb{Z} \times \mathbb{Z} / 2\mathbb{Z})^F_2 \) has an approximate lift if and only if \( \phi^{\text{edge}} \) is (close to) a coboundary.
The Ornstein-Weiss example revisited

The *increase in entropy* corresponds with the exponential growth of the cohomology of the graphs \((V_n, E_n)\).
The Ornstein-Weiss example revisited

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Aside: the Ornstein-Weiss map has been generalized by Gaboriau-Seward from the free group \(\mathbb{F}\) to an arbitrary group \(\Gamma\) and the increase in entropy is related to the first \(\ell^2\)-betti number and cost of \(\Gamma\).
What is this good for?

Gottschalk’s Surjunctivity Conjecture (1973): Let $k$ be a finite set and
$\Phi : k^\Gamma \to k^\Gamma$ a continuous shift-equivariant injective map. Then $\Phi$ is surjective.
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**Theorem (Gromov, 1999)**

*If $\Gamma$ is sofic then the conjecture is true.*
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Theorem (Gromov, 1999)

If $\Gamma$ is sofic then the conjecture is true.

Proof by Kerr-Li, 2011.

- $h_\Sigma(\Gamma \bowtie k^\Gamma) = \log |k|$.
- $h_\Sigma(\Gamma \bowtie \Phi(k^\Gamma)) = \log |k|$.
- The sofic entropy of any proper subshift of $k^\Gamma$ is $< \log |k|$.
Partial actions

We don’t actually need $\Gamma \curvearrowright_{\sigma_n} V_n$ to be actions.
Partial actions

We don’t actually need $\Gamma \curvearrowright_{\sigma_n} V_n$ to be actions.

Instead we require $\{\sigma_n : \Gamma \to \text{sym}(V_n)\}$ to be a sequence of maps (not necessarily homomorphisms!) satisfying

- **(asymptotic homorphism condition)** $\forall g, h \in \Gamma,$
  
  $$|V_n|^{-1} \# \{ v \in V_n : \sigma_n(gh)v = \sigma_n(g)\sigma_n(h)v \} \to 1 \text{ as } n \to \infty$$

- **(asymptotic freeness)** $\forall g \in \Gamma \setminus \{1_{\Gamma}\},$
  
  $$|V_n|^{-1} \# \{ v \in V_n : \sigma_n(g)v \neq v \} \to 1 \text{ as } n \to \infty.$$
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- (asymptotic homomorphism condition) $\forall g, h \in \Gamma$,

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- (asymptotic freeness) $\forall g \in \Gamma \setminus \{1\}$,

$$|V_n|^{-1}\#\{v \in V_n : \sigma_n(g)v \neq v\} \to 1 \text{ as } n \to \infty.$$

Such a sequence is a sofic approximation and $\Gamma$ is sofic if it has one.

Definition due to Gromov (1999), named and made accessible by Weiss (2000).
An action of $\mathbb{Z}^2$
A partial action of $\mathbb{Z}^2$
Sofic groups

- Amenable groups are sofic.
Sofic groups

- Amenable groups are sofic.
- Residually finite groups are sofic. Hence all linear groups are sofic.
Sofic groups

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- Residually finite groups are sofic. Hence all linear groups are sofic.

- The class of sofic groups is closed under: subgroups, direct limits, inverse limits, direct products, extensions by amenable groups, free products with amenable amalgamation, wreath products. (Elek-Szabo, Dykema-Kerr-Pichot, Paunescu, Hayes-Sale)

If $G$ is sofic then $G$ satisfies Gottshalk's surjunctivity conjecture, Connes embedding conjecture, the Determinant conjecture, Kaplansky's direct finiteness conjecture. (Gromov 1999, Weiss 2000, Elek-Szabo 2005)

Open: Is every countable group sofic?
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- **Open**: Is every countable group sofic?
The measure-conjugacy problem

Probability-measure-preserving actions

$$\Gamma \curvearrowright (X, \mu_X), \quad \Gamma \curvearrowright (Y, \mu_Y)$$

are **measurably conjugate** if there exists a measure-space isomorphism $\Phi$

![Diagram](attachment://diagram.png)

intertwining the actions.
The measure-conjugacy problem

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are **measurably conjugate** if there exists a measure-space isomorphism \( \Phi \)

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\end{array}
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intertwining the actions.

**Main Problem:** Classify actions of \( \Gamma \) up to measure conjugacy.
Bernoulli shifts

- If \( \kappa \) is a probability measure on a space \( K \) then the full shift-action

\[ \Gamma \curvearrowright (K^\Gamma, \kappa^\Gamma) \]

with the product measure \( \kappa^\Gamma \) is the **Bernoulli shift** over \( G \) with base \( \kappa \).
Bernoulli shifts

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with the product measure $\kappa^\Gamma$ is the **Bernoulli shift** over $G$ with base $\kappa$.

- **von Neumann’s question**: Is the full 2-shift over $\mathbb{Z}$ measurably conjugate to the full 3-shift?

\[
\begin{align*}
\{0, 1\}^\mathbb{Z} &\xrightarrow{\exists \Phi?} \{0, 1\}^\mathbb{Z} \\
\{0, 1, 2\}^\mathbb{Z} &\xrightarrow{\exists \Phi?} \{0, 1, 2\}^\mathbb{Z}
\end{align*}
\]
Let $\Gamma \curvearrowright (X, \mu)$ be an action by homeomorphisms and $\mu$ an invariant probability measure.
Measure entropy ala Kerr-Li

Let $\Gamma \curvearrowright (X, \mu)$ be an action by homeomorphisms and $\mu$ an invariant probability measure.

The **measure sofic entropy of** $\Gamma \curvearrowright (X, \mu)$ is the exponential growth rate of the number of approximately equidistributed periodic orbits that can be distinguished at scale $\epsilon$ (and then send $\epsilon \downarrow 0$).
Measure entropy ala Kerr-Li

The **empirical distribution** of a map $\phi : V \rightarrow X$ is the probability measure

$$P_\phi := \frac{1}{|V|} \sum_{v \in V} \delta_{\phi(v)} \in \text{Prob}(X).$$
The **empirical distribution** of a map $\phi : V \to X$ is the probability measure

$$P_\phi := \frac{1}{|V|} \sum_{v \in V} \delta_{\phi(v)} \in \text{Prob}(X).$$

If $\mathcal{O} \subset \text{Prob}(X)$ is an open neighborhood of $\mu$ then a $(\sigma, \delta, F, \mathcal{O})$-**pseudo-periodic orbit** is a map $\phi : V \to X$ such that $\phi$ is a $(\sigma, \delta, F)$-pseudo-periodic orbit and $P_\phi \in \mathcal{O}$. 

Lewis Bowen  (UT Austin)  A brief introduction to sofic entropy theory  43 / 47
The **sofic entropy of** $\Gamma \curvearrowright (X, \mu)$ **with respect to** $\Sigma$ **is**

$$h_\Sigma(\Gamma \curvearrowright (X, \mu)) := \sup_{\epsilon > 0} \inf_{\delta, F, \mathcal{O}} \limsup_{n \to \infty} |V_n|^{-1} \log \text{cov}_\epsilon((\sigma, \delta, F, \mathcal{O})\text{-pseudo-periodic orbits}, \rho_\infty).$$
Main Results

1. (Variational Principle, Kerr-Li) $h_\Sigma(\Gamma \curvearrowright X) = \sup_\mu h_\Sigma(\Gamma \curvearrowright (X, \mu))$. 

2. Measure sofic entropy is a measure conjugacy invariant.

3. If $\Gamma$ is amenable then sofic entropy agrees with classical entropy.

4. The sofic entropy of a Bernoulli shift action is the Shannon entropy of the base:

$$h_\Sigma(\Gamma \curvearrowright (K, \kappa)) = H(\kappa) := \sum_{k \in K} -\kappa(\{k\}) \log \kappa(\{k\})$$

5. So the 2-shift over $F_2$ is not isomorphic to the 4-shift over $F_2$!
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Classification of Bernoulli shifts

Conjecture: Assume $|\Gamma| = \infty$. Then

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- $\Gamma$ amenable groups (folklore or Kieffer?, 1970s)
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$\Leftarrow$

- $\Gamma = \mathbb{Z}$ (Ornstein, 1970)
- $\Gamma$ amenable (Ornstein-Weiss, 1980)
- $\mathbb{Z} \leq \Gamma$ (Stepin, 1975)
- $\forall \Gamma$, $|K| > 2$ and $|L| > 2$ (B. 2012)
- $\forall \Gamma$ (Seward, 2018)
Further topics

- Markov chains over free groups (B. 2010): a variant of sofic entropy theory via random permutations yields exact computations and structural results although a classification remains elusive.
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- **Algebraic dynamics**: there is an explicit formula for the sofic entropy of principal algebraic actions and many nice structural results are due to Ben Hayes that generalizes earlier work of D. Kerr, H. Li, B., Deninger-Schmidt, Deninger, Lind-Schmidt-Ward, Rufus Bowen, Yuzvinskii and others.

- Rokhlin entropy is an upper bound for sofic entropy. Brandon Seward has used it to prove generalizations of Krieger's generator theorem and Sinai's Factor Theorem for all countable groups.

- Weak Pinsker Conjecture: Tim Austin recently posted a solution for actions of amenable groups. I have a counterexample in the case of free group actions based on sofic entropy theory and the hardcore model on random regular graphs.
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