Group Theory problems

September 24, 2019

1. Prove that the group G of orientation preserving isometries of $\mathbb{R}^3$ that preserve a regular dodecahedron is isomorphic to A_5. Hint: first determine $|G|$. How many 5-Sylow subgroups does G have?

2. Let n be an odd number so that $\pi = (1, 2, \ldots, n) \in A_n$. Is the S_n-conjugacy class of π the same as its A_n-conjugacy class?

3. Let H be a subgroup of G. Recall that $C_G(H) \lhd N_G(H)$ denotes the centralizer and normalizer of H in G. Show there exists an injective homomorphism from $N_G(H)/C_G(H)$ into $\mathrm{Aut}(H)$.

4. Recall that $Q_8 = \{1, -1, i, j, k, -i, -j, -k\}$ is the group with $ij = k, jk = i, ki = j, i^2 = j^2 = k^2 = -1$. Prove that Q_8 is not isomorphic to a subgroup of S_7. (By contrast, D_8 is isomorphic to a subgroup of S_4).

5. (a) Let M be a non-normal maximal subgroup of a finite group G. Show: the number of elements $g \in G$ that are contained in a conjugate of M is at most $(|M| - 1)[G : M] + 1$.

   (b) Let H be a proper subgroup of G. Show that $G \neq \cup_{g \in G} gHg^{-1}$. Hint: use the previous problem.

   (c) Show that every $M \in GL(n, \mathbb{C})$ is conjugate to an upper triangular matrix. Hint: M has a nonzero eigenvector. Thus the finiteness assumption in the previous problem is necessary.

6. A group is called an elementary abelian p-group if it is isomorphic to $(\mathbb{Z}/p)^n$ for some n. Suppose G is a solvable finite group.

   - Prove G has a nontrivial characteristic abelian subgroup.
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   - Prove there is a nontrivial homomorphism $\phi : \mathrm{Aut}(G) \to GL(n, F_p)$ for some prime p.

7. Show there are exactly 4 homomorphisms from $\mathbb{Z}/2$ into $\mathrm{Aut}(\mathbb{Z}/8)$. Using these, construct the semi-direct products $\mathbb{Z}/8 \rtimes \mathbb{Z}/2$. Show these 4 groups are pairwise non-isomorphic.

8. Let G be a group of order p^k for some prime p and $k \geq 1$. Show that for every $1 \leq l \leq k$ that G has a normal subgroup of order p^l.

9. Show that if $|G| = 336$ then G is not simple.

10. Prove that if $|G| = 231$ then ZG contains a Sylow 11-subgroup.

11. Let n_p be the number of Sylow p-subgroups. Show that if $n_p \neq 1 \mod p^2$ then there exist distinct Sylow p-subgroups P, Q such that $[P : P \cap Q] = p$.

12. Find the ascending and descending central series of S_4.

13. Prove that $\mathbb{R}^2 \rtimes \mathbb{R}^{>0}$ is solvable where $(\mathbb{R}^{>0}, \times)$ acts on $\mathbb{R}^2$ by $t(x, y) = (tx, y/t)$. Also prove this group is not nilpotent. Context: this group is called **SOL**. It represents one of the 8 geometries in the Geometrization Theorem for 3-manifolds.

14. Let p be an odd prime and suppose $|G| = p^3$.

   - Show that the map $\phi : G \to G$ defined by $\phi(g) = g^p$ is a homomorphism from G into the center of G.
   - Suppose all nontrivial elements of G have order p. Show G splits as a semi-direct product.
   - Suppose $x \in G$ has order p^2. Show $N = \langle x \rangle$ is normal in G. Let $y \in \mathrm{Ker}(\phi) - N$. Let $H = \langle y \rangle$. Show $G \cong N \rtimes H$.