Practice problems for M380C Midterm II

November 3, 2019

1. Let R be a commutative ring.

   (a) Show that if $M \subset R$ is a maximal ideal and $1 \in R$ then R/M is a field. (I know this is a theorem in the book. Please prove it from scratch).

   (b) Give an example of a commutative ring R with a maximal ideal $M \subset R$ such that R/M is not a field.

2. An element of a ring is called nilpotent if some power of it is equal to zero.

   (a) Let R be a commutative ring. Show that its nilpotent elements form an ideal. (This ideal is called R's nilradical and written $\mathbf{nil}(R)$.)

   (b) Continuing (a), show that the nilradical of $R/\mathbf{nil}(R)$ is the zero ideal.

   (c) Show that in a noncommutative ring, the nilpotent elements need not form an ideal.

3. (Chinese Remainder Theorem for Modules) Let M be an R-module and $A_1, \ldots, A_n \subset R$ be ideals. Prove that the map

$$M \to M/(A_1M \times \cdots \times A_nM), \quad m \mapsto (m + A_1M, \ldots, m + A_nM)$$

is an R-module homomorphism with kernel $A_1M \cap \cdots \cap A_nM$. Moreover, if the ideals $A_1, \ldots, A_n$ are pairwise co-maximal then

$$M/(A_1 \cdots A_n)M \cong M/A_1M \times M/A_kM.$$

4. Prove or disprove: if R is a UFD and $I \subset R$ is a prime ideal then R/I is a UFD.

5. Let R be an integral domain. Prove that R is a UFD if and only if every prime ideal $I \subset R$ contains a prime element $x \in I$.

6. Let R be a PID. An ideal $I \subset R$ is **primary** if for all $a, b \in R$ with $ab \in I$ either $a \in I$ or there exists $n \in \mathbb{N}$ such that $b^n \in I$. Prove that if $I \subset R$ is primary then there exists a prime element $p \in R$ and $n \in \mathbb{N}$ such that $I = (p^n)$.

7. (a) Let R be a PID. Prove that every prime ideal is maximal.

   (b) Does $\mathbb{Z}[x]$ have a prime ideal that is not maximal? If so, give an example. If not, prove it.

8. Let R be an integral domain. Suppose the following conditions hold.

   - Any two elements a, b have a gcd d that is a linear combination of a, b (so $d = ax + by$ for some $x, y \in R$).
   - If $a_1, a_2, \ldots \in R$ are such that $a_{i+1} \mid a_i$ for all i then there is an N such that for every $n, m > N$, a_n is an associate of a_m.

   Prove that R is a PID.

9. Prove that irreducible elements are prime. Also show that if R is a PID then prime elements are irreducible.

10. Prove that PID's are UFD's. You may use the fact that, in a PID, an element is prime if and only if it is irreducible.

11. Let R be a Noetherian domain and $r \in R$ be a non-unit. Prove that there exist irreducible elements $p_1, \ldots, p_n$ such that $r = p_1 \cdots p_n$.

12. Find the gcd of $x^3 - 2$ and $x + 1$ in $\mathbb{Q}[x]$ and write it as a linear combination of $x^3 - 2$ and $x + 1$.

13. Determine all integer solutions to $85x + 145y = 505$.

14. Prove that a polynomial $p \in \mathbb{Z}[x]$ is irreducible in $\mathbb{Z}[x]$ if and only if it is irreducible in $\mathbb{Q}[x]$ and its content is 1. (Its content is the gcd of all of its coefficients). You can use Gauss' Lemma.

15. State and prove Gauss' Lemma.

16. Determine all the ideals of the ring $R = \mathbb{Z}[x]/(2, x^3 + 1)$.

17. Let $f \in \mathbb{Z}[x]$ be irreducible in $\mathbb{Z}[x]$. Prove that if $I \subset \mathbb{Z}[x]$ is a maximal ideal with $(f) \subset I$ then $I = (f, p)$ for some integer prime p.

18. Prove or disprove:

    (a) $\mathbb{R}[x, y]$ is a Euclidean domain.

    (b) $\mathbb{R}[x, y]$ is Noetherian.

    (c) $\mathbb{Z}[x]$ is a principal ideal domain (PID).

    (d) $\mathbb{C}[x, y]/(y^2 - 2x - 3)$ is a Euclidean Domain.

    (e) $\mathbb{C}[x]/(x^2 + 1)$ is a field.

    (f) $\mathbb{R}[x]/(x^2 + 1)$ is a field.

    (g) $\mathbb{Q}[x]/(x^2 + 1)$ is a field.

    (h) $\mathbb{Z}[x]/(x^2 + 1)$ is a field.

    You can use any of the Theorems we discussed in class.

19. Let F be a field of characteristic p and let $c \in F$. Let $f(x) = x^p - c$. Suppose that f has no roots in F. Show that f is irreducible in $F[x]$. Hint: you may use the fact that there exists some field $K \supset F$ such that f splits over K in the sense that $f(x) = (x - z_1) \cdots (x - z_p)$ for some elements $z_1, \ldots, z_p \in K$.