M380C Fall 2019, Memory Jogger for Exam 2

Instructions: Put away your book and notes and try to answer the following questions from memory. If you struggle with a question, circle it. After you're done with this list, pick up the textbook and try to answer your circled questions.

1. There are special kinds of rings such as commutative/non-commutative rings, integral domains, fields, division rings, algebras. Explain what these are and provide some examples.
2. There are special types of elements in rings such as zero divisors, units, nilpotent elements and idempotents. Explain what these are and provide some examples.
3. There are special kinds of subsets of rings such as ideals (left, right and 2-sided) and sub-rings. Explain and provide examples. What are some examples of infinite rings that have no proper 2-sided ideals? Give commutative and non-commutative examples.
4. Explain ring homomorphisms, kernels, quotients and the isomorphism theorems for rings. Provide some examples.
5. What is the relationship between prime ideals, maximal ideals, and integral domains and fields?
6. Given ideals $A, B \subset R$, explain $A + B, A \cap B, AB$, the ideal generated by A and B. What are the relationships between these objects?
7. Given a subset $D \subset R$, when is $D^{-1}R$ well-defined? What does it mean? Provide some examples.
8. Explain the Chinese Remainder Theorem and some applications. Outline the proof.
9. There are special kinds of elements in integral domains: prime, irreducible, associates, gcds. Explain these, provide examples and explain the relationships between them.
10. Interpret the concepts above (prime, irreducible, associates, gcds) in terms of ideals.
11. There are special kinds of integral domains: FD's, UFD's, PID's, Euclidean domains, GCD domains, Noetherian domains. Explain these, provide examples and explain the relationships between them. Outline the proofs.
12. What is a quadratic integer ring? Give some examples.
13. Explain prime factorization in $\mathbb{Z}[i]$ and which primes are sums of squares.
14. Prove Gauss's Lemma. Then prove: if R is a UFD then $R[x]$ is a UFD.
15. Prove or disprove: a polynomial $p \in \mathbb{Z}[x]$ is irreducible in $\mathbb{Z}[x]$ if and only if it is irreducible in $\mathbb{Q}[x]$.
16. Explain Eisenstein's criterion and some other criteria for irreducibility.

17. Prove: if F is a finite field then $F^\times$ is cyclic.
18. What's a Noetherian ring again? Does it have something to do with the ascending chain condition or the descending chain condition?
19. The Hilbert Basis Theorem states that if R is Noetherian then so is $R[x]$. Prove it. What does this imply about $\mathbb{Z}[x_1, \ldots, x_n]$?