M380C Fall 2019, Memory Jogger for the Final

Instructions: Put away your book and notes and try to answer the following questions from memory. If you struggle with a question, circle it. After you're done with this list, pick up the textbook and try to answer your circled questions.

1. Define R-modules, R-module homomorphisms, submodules, quotient modules. Recall the isomorphism theorems for modules.

2. Define free R-modules and explain the universal property of free modules. Suppose R is commutative. Prove that $R^n \cong R^m$ (as R-modules) if and only if $m = n$.

3. Define multilinear, alternating and symmetric maps.

4. Tensor products.

   (a) Define $S \otimes_R M$ where $R \subset S$ are rings and M is a left R-module. Go through some examples such as $\mathbb{Q} \otimes_{\mathbb{Z}} (\mathbb{Q}/\mathbb{Z})$, $\mathbb{Z}^2 \otimes_{\mathbb{Z}} \mathbb{Z}/n$, $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{R}^2$, $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Z}[i]$. Recall the universal property.

   (b) Define $M \otimes_R N$ where M is a left R-module and N is a right R-module. Recall the universal property.

   (c) Recall that if R is commutative and M, N are R-modules, then $M \otimes_R N$ has an R-module structure. Work through some examples and recall the universal property.

   (d) Suppose M, N are R-algebras. Show that $M \otimes_R N$ has a natural R-algebra structure.

   (e) Prove that $(A \otimes B) \otimes C \cong A \otimes (B \otimes C)$.

   (f) Prove that $(A \oplus B) \otimes C \cong (A \otimes C) \oplus (B \otimes C)$.

   (g) If $f : A \to C$ and $g : B \to D$ are R-module homomorphisms, then what is $f \otimes g$?

5. Prove that if F is a field then and V is a finite-dimensional vector space over F then any two maximal linearly independent subsets of V have the same cardinality.

6. Define the determinant $\det : M_{n\times n}(R) \to R$ where R is an arbitrary commutative ring with 1. What does it have to do with alternating maps? Prove that $\det(AB) = \det(A)\det(B)$.

7. Prove the cofactor expansion theorem for computing the determinant.

8. Define the tensor algebra $\mathcal{T}(M)$ of an R-module M. Define the symmetric tensor algebra $\mathcal{S}(M)$ and the exterior algebra $\bigwedge(M)$. If $R = F$ is field and $M = V$ is an n-dimensional vector space then give a basis for $\mathcal{T}^k(V), \mathcal{S}^k(V), \wedge^k(M)$. What are the universal properties of these algebras?

9. If $f : V \to V$ is linear, what is $\wedge^k f$? Pay special attention to the case $k = n$ when V is n-dimensional.

10. Define projective modules, short exact sequences that split, direct factors of free modules. What do these things have to do with each other? What are the projective $\mathbb{Z}$-modules? What are the projective $\mathbb{Z}/n\mathbb{Z}$-modules?

11. Suppose that R is a PID, M is a free R-module of finite rank and $M' \subset M$ is a submodule. Prove that M' is also free and rank$(M') \leq$rank(M) (without using the 'big' theorem). Hint: If M is freely generated by $x_1, \ldots, x_n$ then let M_i be the submodule generated by $x_1, \ldots, x_i$. Consider $M' \cap M_i$.

12. Assume R is a PID. Prove that if M is a torsion-free finitely generated R-module then M is free (without using the 'big' theorem). Hint: M contains a 'maximal' free submodule M' and there is an injective map $\phi : M \to M'$. Construct ϕ then apply the previous problem.

13. Assume R is a PID and M is a finitely generated R-module. Prove that $M \cong F \oplus M_{tors}$ where F is a free module and M_{tors} is the torsion submodule.

14. If $p \in R$ is prime then $Mp := \{x \in M : \exists n \in \mathbb{N}, p^n x = 0\}$. Prove that $M = \oplus_p M_p$.

15. Explain the 'big theorem' for finitely generated modules over PIDs. What are the elementary divisors? The invariant factors? Explain how to derive the elementary divisors from the invariant factors and vice versa.

16. Suppose R is a PID and $p \in R$ is prime. Without using the 'big theorem', prove that if

$$(R/(p))^{n_1} \oplus (R/(p^2))^{n_2} \cong (R/(p))^{m_1} \oplus (R/(p^2))^{m_2}$$

then $n_1 = m_1$ and $n_2 = m_2$. Work out a similar problem with three summands.

17. Let V be a finite-dimensional vector space over a field F. Let $T : V \to V$ be linear. How is V an $F[x]$-module? Why is it torsion?

18. Define the characteristic and minimal polynomials of T. What do they have to with the elementary divisors and/or the invariant factors?

19. What is rational canonical form? How do you choose the basis?

20. What is Jordan canonical form? How do you choose the basis?

21. Define the Zariski topology. Prove that it really is a topology.

22. We gave two equivalent definitions of a radical ideal. Prove that they really are equivalent.

23. Define irreducibility (of an algebraic affine set). Prove that V is irreducible if and only if $\mathcal{I}(V)$ is prime.

24. Prove that any affine algebraic set V can be written uniquely as $V = \cup_i V_i$ with V_i irreducible and $V_i \nsubseteq V_j$ if $i \neq j$.

25. Let $1 \in R \subset S$ be commutative rings. Let $s \in S$. Define what it means for s to be integral in R. Define the integral closure of R in S. What's the connection with monic polynomials and finitely generated somethings?

26. Prove that the integral closure of R in S is a ring.

27. State Noether's Normalization Lemma. Use it to prove Hilbert's Nullstellensatz (at least the weak version).

28. Define Noetherian ring. In fact, there are several equivalent definitions. Does it have to do with the ascending or descending chain condition? Prove that if R is Noetherian and $\mathcal{I}$ is a collection of ideals in R then there exists an ideal I in $\mathcal{I}$ which is maximal amongst all ideals in $\mathcal{I}$ (ordered by inclusion).