A topological dynamical system with two different positive sofic entropies

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Overview

- Sofic groups
- Sofic entropy
- Main theorem
- Proper 2-colorings of random hyper-graphs

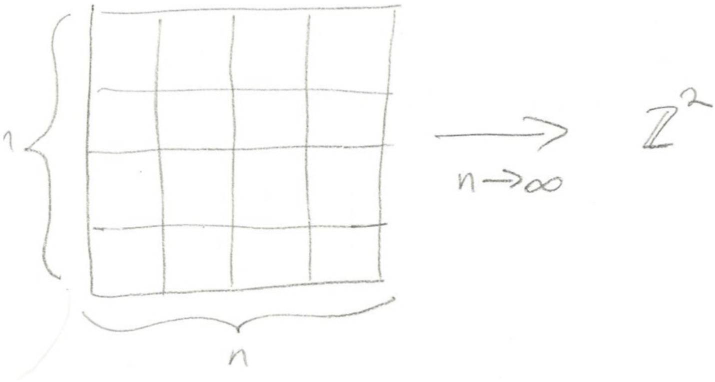
Benjamini-Schramm convergence

Definition

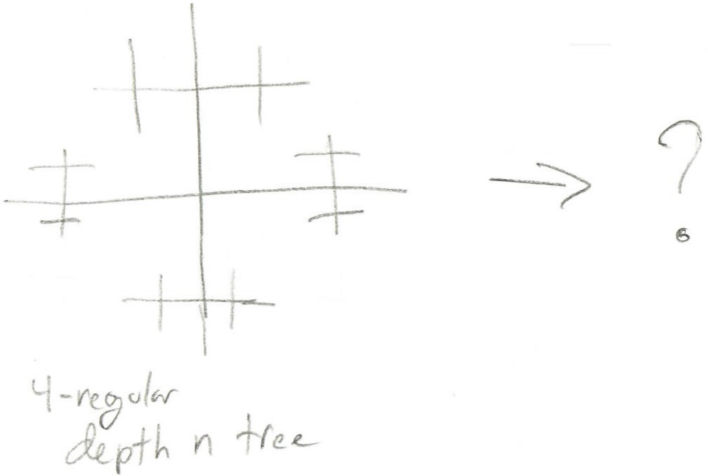
A sequence $G_i = (V_i, E_i)$ of finite graphs **BS-converges** to a fixed vertex-transitive graph $G = (V, E)$ if for every radius $r > 0$,

$$\lim_{i\to\infty} \frac{\#\{v \in V_i : \ B_r(G_i, v) \cong B_r(G, o)\}}{\# V_i} = 1.$$

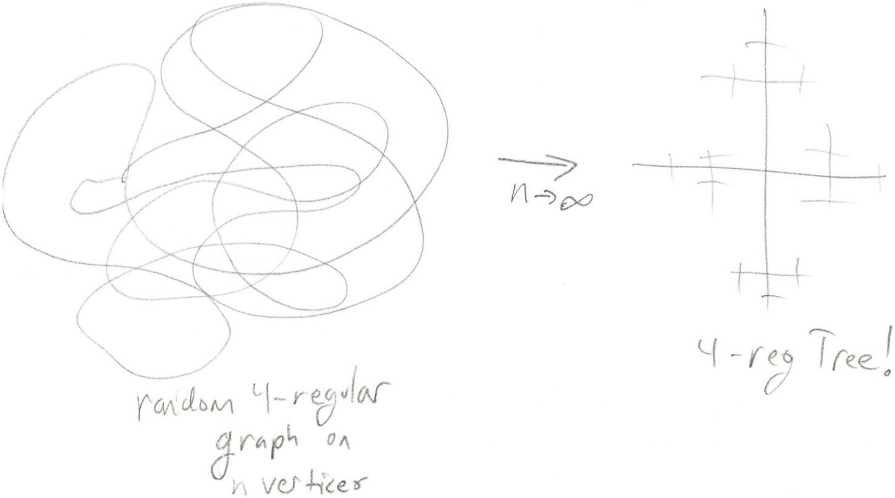
The square grid as a BS-limit



?


4-regular
depth n tree

4-regular tree



Sofic groups

Let Γ be a countable group with finite generating set S.

The **Cayley graph** of (Γ, S) has vertex set $V = \Gamma$ and edges

$$E = \{(g, gs) : \; g \in \Gamma, s \in S\}.$$

A **sofic approximation** to Γ is a sequence $\Sigma = \{G_i = (V_i, E_i)\}$ of finite graphs that BS-converges to the Cayley graph.

Γ is **sofic** if it has a sofic approximation.

Question

Are all countable groups sofic?

(and now for something completely different)

Theorem (B.-Burton, 2019)

For $d \geq 5$, either $PSL(d, \mathbb{Z})$ is not flexibly stable or there exists a non-sofic group.

Sofic entropy (overview)

Given a continuous action $\Gamma \curvearrowright X$, the **$\Sigma$-entropy** (or **sofic entropy**) is

$$h_\Sigma(\Gamma \curvearrowright X) \in \{-\infty\} \cup [0, \infty].$$

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Main properties

1. Monotone under embeddings:

$$Y \subset X \Rightarrow h_\Sigma(\Gamma \curvearrowright Y) \leq h_\Sigma(\Gamma \curvearrowright X).$$

2. $h_\Sigma(\Gamma \curvearrowright A^\Gamma) = \log |A|$.
3. Γ amenable $\Rightarrow h_\Sigma(\Gamma \curvearrowright X) =$ classical topological entropy (Kerr-Li, 2012).
4. There is a measure entropy version (B. 2010) and a variational principle (Kerr-Li, 2011).

Symbolic dynamics

Let

- A be a finite 'alphabet',
- $A^\Gamma = \{x : \Gamma \to A\}$,
- $\Gamma \curvearrowright A^\Gamma$ by $(g \cdot x)_f = x_{g^{-1}f}$.
- $X \subset A^\Gamma$ be closed and Γ-invariant. X is a **subshift**.

The entropy of a subshift

Let $\Sigma = \{G_n = (V_n, E_n)\}$ be a sofic approximation.

A coloring $\chi : V_n \to A$ is an **(r,ϵ)-model** for $\Gamma \curvearrowright X$ on G_n if there exists a subset $W_n \subset V_n$ such that

- $|W_n| \geq (1-\epsilon)|V_n|$,
- for all $v \in W_n$, there exists $x \in X$ such that $B_r(G_n, \chi, v) \cong B_r(\Gamma, x, 1_\Gamma)$.

Definition

$$h_\Sigma(\Gamma \curvearrowright X) := \inf_{r>0} \inf_{\epsilon>0} \limsup_{n\to\infty} |V_n|^{-1} \log \#\{(r,\epsilon)\text{-models for } \Gamma \curvearrowright X \text{ on } G_n\}.$$

Question

How does Σ-entropy depend on Σ?

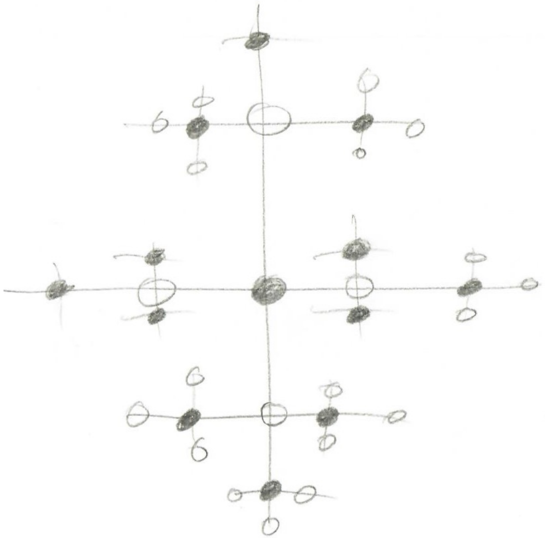
A degenerate example

Let

- $\Gamma = \langle a, b \rangle$,
- $X = \{x, y\} \subset \{0, 1\}^{\Gamma}$ be the two bi-partite colorings of Γ,
- $\Sigma = \{G_n\}$ be a sofic approximation to the Cayley graph.

- If each G_n is bi-partite then $h_{\Sigma}(\Gamma \curvearrowright X) = 0$.
- If G_n is 'random' then $h_{\Sigma}(\Gamma \curvearrowright X) = -\infty$.

A degenerate example



The main result

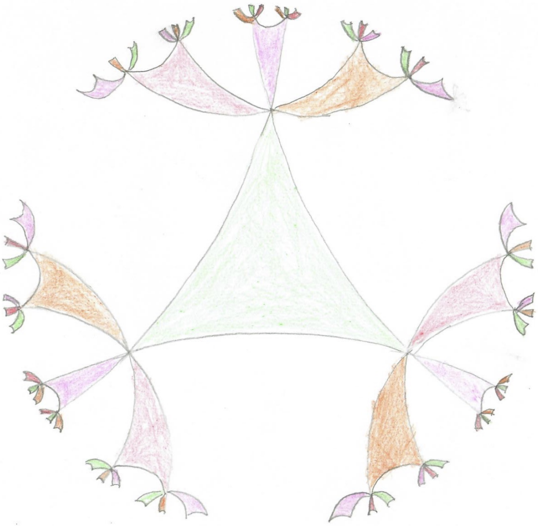
Theorem (Airey-B.-Lin)

There exists an explicit group action $\Gamma \curvearrowright X$ *and sofic approximations* Σ_1, Σ_2 *such that*

$$0 < h_{\Sigma_1}(\Gamma \curvearrowright X) < h_{\Sigma_2}(\Gamma \curvearrowright X).$$

The measure entropy case is still open.

The example



$$\Gamma = \langle s_1, \ldots, s_d : \; s_1^k = s_2^k = \cdots = s_d^k = 1 \rangle.$$

Hyper-graphs and proper colorings

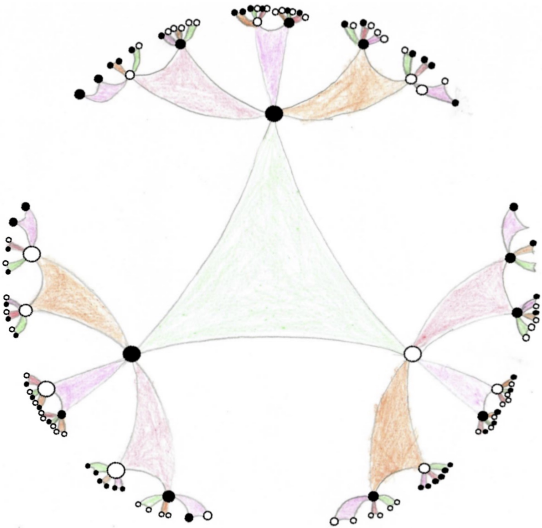
A **hyper-graph** is a pair $G = (V, E)$ where E is a collection of subsets of V.

A coloring $\chi : V \to \{0, 1\}$ is **proper** if $\chi(e) = \{0, 1\}$ for every $e \in E$.

The **Cayley hyper-graph** of Γ has $V = \Gamma$ and $E = \{$left-cosets of generator subgroups$\}$.

Let $\mathbf{X} \subset \{0, 1\}^\Gamma$ be the subset of proper colorings of the Cayley hyper-graph.

A proper coloring



Sofic approximations

Fix a sofic approximation $\Sigma = \{G_n = (V_n, E_n)\}$ to Γ.

A coloring $\chi : V_n \to \{0, 1\}$ is **ϵ-proper** if

$$\#\{e \in E_n : |\chi(e)| = 1\} \le \epsilon n.$$

If Σ is any sofic approximation then

$$h_\Sigma(\Gamma \curvearrowright X) = \inf_{\epsilon>0} \limsup_{n\to\infty} n^{-1} \log \#\{ \epsilon\text{-proper colorings of } G_n\}.$$

Random sofic approximations

Suppose V_n is a sequence of finite sets and $\mathbb{P}_n$ are probability measures on the set of hyper-graphs on V_n. Then $\mathbb{P}_n$ is a **random sofic approximation** if for every $r > 0$ and $\delta > 0$ there exists $\epsilon > 0$ such that

$$\mathbb{P}_n\left(\frac{\#\{v \in V_n : \ B_r(G_n, v) \cong B_r(G, o)\}}{\#V_n} > 1 - \delta\right) > 1 - n^{-\epsilon n}$$

for all large enough n.

Main observation

If $\mathbb{P}_n$ is a random sofic approximation and Ω_n are collections of hyper-graphs with $\mathbb{P}_n(\Omega_n) \geq e^{-cn}$ for some $c > 0$ then $\exists$ a sofic approximation $\Sigma = \{G_n\}$ with $G_n \in \Omega_n$ for all n.

The uniform model

Given a homomorphism $\sigma : \Gamma \to \mathrm{sym}(n)$, let G_σ be the hyper-graph on $[n]$ with edges being the orbits of the generators:

$$E_\sigma = \left\{ \left\{ \sigma(s_i^j)v \right\}_{j=0}^{k-1} : \; i \in [d], v \in [n] \right\}.$$

σ is a **uniform** if G_σ is ***k*-uniform** .

The **uniform model** is the uniform probability measure $\mathbb{P}_n^{\mathrm{unif}}$ on the set of uniform homomorphisms. By abuse, we also say the **uniform model** is the induced sequence of distributions on hyper-graphs. It is a random sofic approximation.

Equitable colorings

Let

- $Z(\epsilon;\sigma) := \#\{\epsilon\text{-proper colorings of } G_\sigma\}$.
- $Z(\sigma) := \#\{\text{proper colorings of } G_\sigma\}$,
- a coloring $\chi : V \to \{0,1\}$ is **equitable** if $|\chi^{-1}(0)| = |\chi^{-1}(1)| = |V|/2$.
- $Z_e(\sigma) := \#\{\text{equitable proper colorings of } G_\sigma\}$.

Then

$$\begin{aligned} \mathbb{E}_n^{\text{unif}}[Z(\epsilon;\sigma)] &\approx e^{O(\epsilon)n}\mathbb{E}_n^{\text{unif}}[Z(\sigma)] \\ \mathbb{E}_n^{\text{unif}}[Z(\sigma)] &\approx \mathbb{E}_n^{\text{unif}}[Z_e(\sigma)]. \end{aligned}$$

The satisfiability threshold

Fix a large k and let $r = d/k$. Conjecturally, there exists a satisfiability threshold $r_{\mathbf{sat}} > 0$ such that

- if $r < r_{\text{sat}}$ then w.h.p, proper colorings exist,
- if $r > r_{\text{sat}}$ then w.h.p, proper colorings do not exist.

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Dimitris Achlioptas and Cristopher Moore.
Random k-SAT: two moments suffice to cross a sharp threshold,
SIAM J. Comput., 36(3):740–762, 2006.

Amin Coja-Oghlan and Lenka Zdeborová.
The condensation transition in random hypergraph 2-coloring,
arXiv:1107.2341, 35 pages, 2011.

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The condensation transition in random hypergraph 2-coloring,
Proceedings of the Twenty-Third Annual ACM-SIAM Symposium on Discrete Algorithms, 241–250. ACM, New York, 2012.

A tale of two moments

Let

- $r_{\mathbf{first}}$:= the value of r with $\lim_{n\to\infty} n^{-1} \log \mathbb{E}_n^{\mathrm{unif}}[Z_e(\sigma)] = 0$.
- $r_{\mathbf{second}}$:= the largest value of r with

$$\lim_{n\to\infty} n^{-1} \log \mathbb{E}_n^{\mathrm{unif}} \left[Z_e^2(\sigma) \right] = \lim_{n\to\infty} n^{-1} \log \mathbb{E}_n^{\mathrm{unif}}[Z_e(\sigma)]^2.$$

Then

$$r_{\mathrm{sat}} \in [r_{\mathrm{second}}, r_{\mathrm{first}}].$$

Theorem (cf. Achlioptas-Moore, 2006)

$$\begin{aligned} r_{first} &= \frac{\log(2)}{2} 2^k - \frac{\log(2)}{2} + O(2^{-k}) \\ r_{second} &= \frac{\log(2)}{2} 2^k - \frac{\log(2)+1}{2} + O(2^{-k}). \end{aligned}$$

The second moment

Lemma

$\mathbb{E}_n^{unif}[Z_e^2] = \mathbb{E}_n^{unif}[Z_e]\mathbb{E}_n^{unif}[Z_e|\chi \textit{ is proper}]$ *where* χ *is any equitable coloring.*

Let $\mathcal{H} = \mathrm{Hom}_{\text{k-uniform}}(\Gamma, \mathrm{sym}(n))$.

$$\begin{aligned}
\mathbb{E}_n^{\text{unif}}[Z_e^2] &= |\mathcal{H}|^{-1} \sum_{\sigma\in\mathcal{H}} \left(\sum_{\chi \text{ equitable}} 1_{(\chi \text{ is proper w.r.t. } G_\sigma)} \right)^2 \\
&= |\mathcal{H}|^{-1} \sum_{\sigma\in\mathcal{H}} \sum_{\chi,\tilde{\chi} \text{ equitable}} 1_{(\chi \text{ is proper w.r.t. } G_\sigma)} 1_{(\tilde{\chi} \text{ is proper w.r.t. } G_\sigma)} \\
&= \sum_{\chi,\tilde{\chi} \text{ equitable}} \mathbb{P}_n^{\text{unif}}(\chi \text{ is proper and } \tilde{\chi} \text{ is proper}) \\
&= \sum_{\chi,\tilde{\chi} \text{ equitable}} \mathbb{P}_n^{\text{unif}}(\chi \text{ is proper })\mathbb{P}_n^{\text{unif}}(\tilde{\chi} \text{ is proper } |\chi \text{ is proper}) \\
&= \mathbb{E}_n^{\text{unif}}[Z_e]\mathbb{E}_n^{\text{unif}}[Z_e|\chi \text{ is proper}].
\end{aligned}$$

The planted model

The **planted model** is $\mathbb{P}_n^{\text{plant}}(\cdot) = \mathbb{P}_n^{\text{unif}}(\cdot|\chi \text{ is proper})$ where χ is any equitable coloring. It is a random sofic approximation.

- $r < r_{\text{second}} \Rightarrow \mathbb{E}_n^{\text{unif}}[Z_e] \approx \mathbb{E}_n^{\text{plant}}[Z_e]$,
- $r_{\text{second}} < r \Rightarrow \mathbb{E}_n^{\text{unif}}[Z_e] \ll \mathbb{E}_n^{\text{plant}}[Z_e]$.

The *enhanced* second moment

Theorem (cf. Coja-Oghlan-Zdeberova, 2011)

For $r \in \left(r_{second}, r_{second} + \frac{1-\log(2)}{2}\right)$,

$$\mathbb{P}_n^{plant}\left(Z_e \leq \mathbb{E}_n^{unif}[Z_e]\right) = \Omega(1/n^C).$$

Therefore,

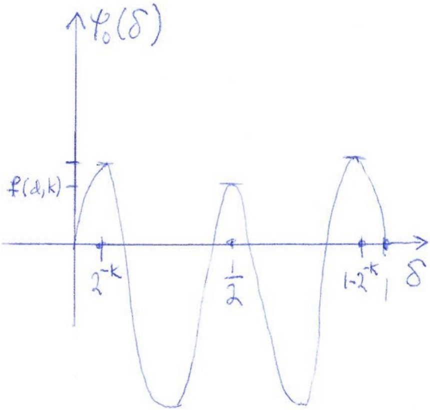
- $\mathbb{P}_n^{unif}\left(Z_e \approx \mathbb{E}_n^{unif}[Z_e]\right) = \Omega(1/n^C)$,
- $\mathbb{P}_n^{plant}\left(Z_e \gg \mathbb{E}_n^{unif}[Z_e]\right) = \Omega(e^{-cn})$,
- $r_{second} < r_{sat}$,
- $\exists \Sigma_1, \Sigma_2$ *such that*

$$0 < h_{\Sigma_1}(\Gamma \curvearrowright X) = f(d,k) < h_{\Sigma_2}(\Gamma \curvearrowright X).$$

Expected number of proper colorings at distance δ from the planted colorings

Theorem

Let $Z_e(\delta)$ be the number of proper colorings at Hamming distance δn from the planted coloring. Then $\mathbb{E}_n^{plant}[Z_d(\delta)] \approx e^{n\psi_0(\delta)}$ where ...



The local cluster

Fix an equitable $\chi : [n] \to \{0,1\}$. This is the **planted coloring**.

The **local cluster** is the set of G_σ-proper equitable colorings $\psi : [n] \to \{0,1\}$ such that $d_{\mathrm{Hamming}}(\chi, \psi) \le 2^{-k/2} n$.

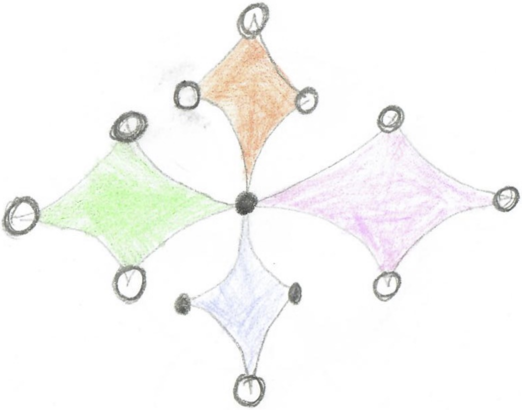
In order to prove

$$\mathbb{P}_n^{\mathrm{plant}} \left(Z_e \le \mathbb{E}_n^{\mathrm{unif}}[Z_e] \right) = \Omega(1/n^C),$$

it suffices to show

$$\mathbb{P}_n^{\mathrm{plant}} \left(|\text{local cluster}| \le \mathbb{E}_n^{\mathrm{unif}}[Z_e] \right) = \Omega(1/n^C).$$

A supporting role



A vertex v **supports** an edge e if $\chi(v) \cap \chi(e \setminus \{v\}) = \emptyset$.

The core

Let $C_0 = [n]$, and C_{l+1} be the set of all $v \in [n]$ such that there are at least three edges e such that

- v supports e,
- $e \setminus \{v\} \subset C_l$.

Note $C_0 \supset C_1 \supset \cdots$.

The **core** is $C_\infty = \cap_l C_l$.

The rigid core

Lemma

With high probability (and a few lies),

- $\frac{|C_\infty|}{n} = 1 - 2^{-k}(1 + o_k(1))$.
- *If* $\psi : [n] \to \{0,1\}$ *is a proper coloring then either*
  - $\psi \restriction C_\infty = \chi \restriction C_\infty$ *or*
  - $\#\{v \in [n] : \ \psi(v) \neq \chi(v)\} \geq \mathit{constant} \cdot \frac{n}{k^4}$.

$\Rightarrow$ whp $\# \left\{ \psi : \ d(\psi, \chi) \leq 2^{-k/2} \right\} \leq 2^{n - |C_\infty|} \leq \mathbb{E}_n^{\mathrm{unif}}[Z_e]$.

Open problems

- Provide general necessary or sufficient conditions (on subshifts of finite type say) for when the sofic entropy range is greater than a singleton.
- The measure entropy case is **wide** open.
- Is there a well-defined notion of mod p ℓ^2-Betti numbers that does not depend on the choice of sofic approximation?

Thank you!