One version of Fubini's theorem for a rectangle says

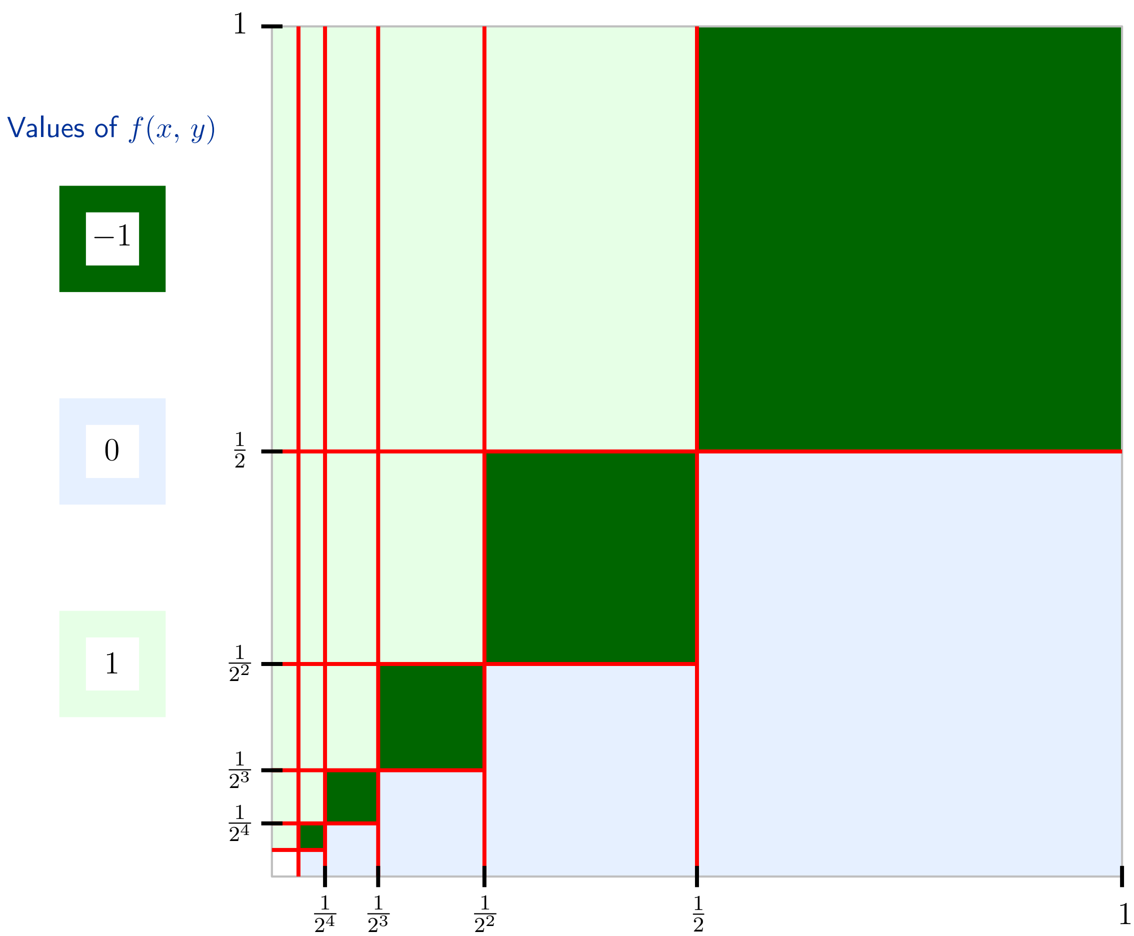
Theorem: if $z = F(x,\, y)$ is continuous on a rectangle $D = [a,\, b] \times [c,\, d]$, then

$$\int_a^b \left(\int_c^d F(x,\, y)\, dy \right) dx \;=\; \int_c^d \left(\int_a^b F(x,\, y)\, dx \right) dy\,.$$

Surprisingly, however, without continuity, the theorem may fail. We shall define a function $z = F(x,\, y)$ on the square $D = [0,\, 1] \times [0,\, 1]$ for which Fubini's Theorem fails in the sense that

$$\int_0^1 \left(\int_0^1 F(x,\, y)\, dy \right) dx \;\neq\; \int_0^1 \left(\int_0^1 F(x,\, y)\, dx \right) dy\,.$$

First we'll define a function $z = f(x,\, y)$ on $[0,\, 1] \times [0,\, 1]$ taking only the values $-1,\, 0$ or 1:



Algebraically,

$$f(x, y) = -1 \quad \text{when} \quad \frac{1}{2^k} < x, y \leq \frac{1}{2^{k-1}}, \quad k \geq 1,$$

while

$$f(x, y) = 1 \quad \text{when} \quad \begin{cases} \dfrac{1}{2^k} < x \leq \dfrac{1}{2^{k-1}} \\ \dfrac{1}{2^m} < y \leq \dfrac{1}{2^{m-1}} \end{cases}, \quad 1 \leq m < k;$$

otherwise $f(x, y) = 0$. Now set

$$F(x, y) = \frac{f(x, y)}{y^2}.$$

For fixed y, $0 < y \leq 1$, the function $f(x, y)$ takes values 1 and -1 on intervals of equal length in x, so

$$\int_0^1 F(x, y)\, dx = \frac{1}{y^2}\int_0^1 f(x, y)\, dx = 0.$$

On the other hand, for $\dfrac{1}{2} < x \leq 1$, we see that

$$\int_0^1 F(x, y)\, dy = \int_0^1 \frac{f(x, y)}{y^2}\, dy = -\int_{1/2}^1 \frac{1}{y^2}\, dy = \left[\frac{1}{y}\right]_{1/2}^1 = -2 + 1 = -1.$$

But now we come to the crucial point of the definition of f. For $\dfrac{1}{2^k} < x \leq \dfrac{1}{2^{k-1}}$ with $k > 1$,

$$\int_0^1 F(x, y)\, dy = \int_{1/2^k}^1 \frac{f(x, y)}{y^2}\, dy = -\int_{1/2^k}^{1/2^{k-1}} \frac{1}{y^2}\, dy + \int_{1/2^{k-1}}^1 \frac{1}{y^2}\, dy = \left[\frac{1}{y}\right]_{1/2^k}^{1/2^{k-1}} - \left[\frac{1}{y}\right]_{1/2^{k-1}}^1$$

$$= \left(2^{k-1} - 2^k\right) - \left(1 - 2^{k-1}\right) = -1 + 2.2^{k-1} - 2^k = -1,$$

because $2^{k-1}.2 = 2^k$. Consequently,

$$\int_0^1 F(x, y)\, dx = 0, \quad 0 < y \leq 1, \qquad \int_0^1 F(x, y)\, dy = -1, \quad 0 < x \leq 1.$$

Thus

$$\int_0^1 \left(\int_0^1 F(x, y)\, dx\right) dy = 0 \neq -1 = \int_0^1 \left(\int_0^1 F(x, y)\, dy\right) dx.$$

What's gone wrong? Well, there's no way $F(x, y)$ can be made continuous at $(0, 0)$ because as (x, y) approaches $(0, 0)$, $F(x, y)$ is blowing up to $+\infty$ in one direction, to $-\infty$ in another direction, and converging to 0 in yet another direction.