A MODEL FOR SOAP FILMS BASED ON CAPILLARITY THEORY

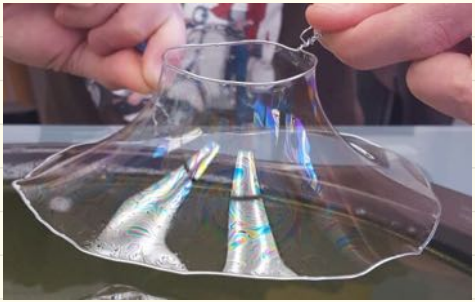
FRANCESCO MAGGI UT AUSTIN

ONE WORLD PDE SEMINAR

MAY 19 2020

JOINT PAPERS WITH ANTONELLO SCARDICCHIO (ICTP)
DARREN KING & SALVATORE STUVARD (UT AUSTIN)

MINIMAL SURFACES = BASIC MODEL FOR SOAP FILMS HANGING FROM A WIRE FRAME AT MECHANICAL EQUILIBRIUM



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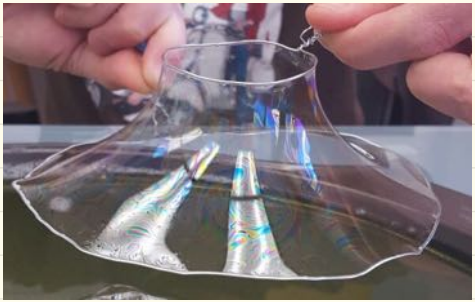

WIRE FRAME W CLOSED CURVE

SOAP FILM S SURFACE

"HANGING FROM" $\partial S = W$

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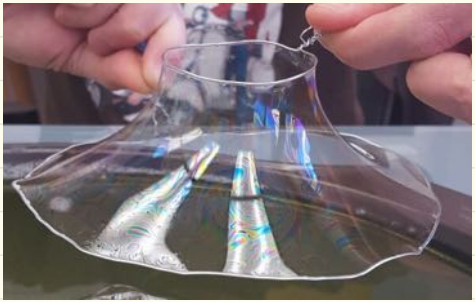

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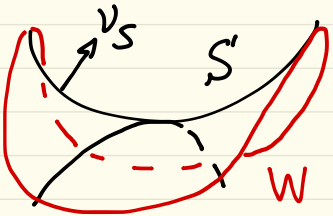
MECHANICAL EQUILIBRIUM $H_S = 0$ ZERO MEAN CURVATURE



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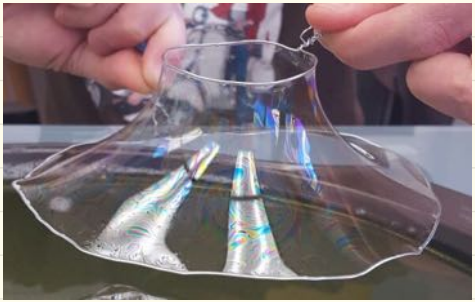

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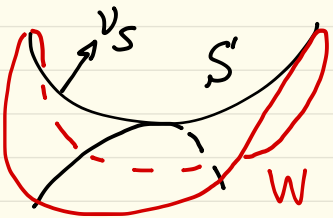

$\longrightarrow$ BALANCE OF PRESSURES

SURFACE TENSION VS ATM. PRESSURE

$\hookrightarrow$ LAPLACE PRESSURE

YOUNG ≈ 1805

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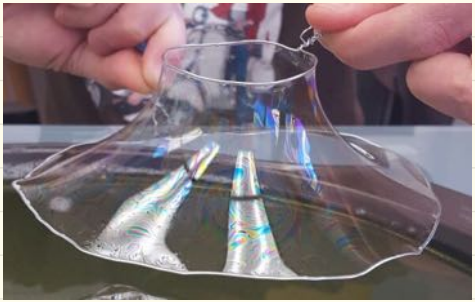

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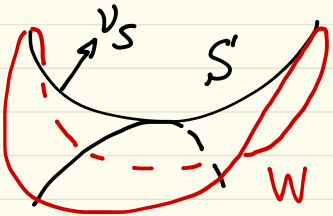


$\longrightarrow$ BALANCE OF PRESSURES

$\longrightarrow$ PRINCIPLE OF VIRTUAL WORK GAUSS 1830

$$\frac{d}{dt}\Big|_{t=0} \text{Area}\left(\text{Id} + t u \nu_{S'}\right) = \int_{S'} u H_{S'}$$

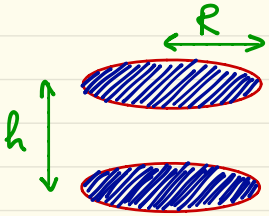
IF $u = 0$ ON W

LIMITATIONS OF THE MODEL $\quad H_S = 0 \quad \partial S = W$

NON-UNIQUENESS $\quad W$ TWO COAXIAL CIRCLES $\quad h/R \lesssim 1.33$

$\Longrightarrow$ THREE DIFFERENT MINIMAL SURFACES S.T. $\partial M = W$

1)



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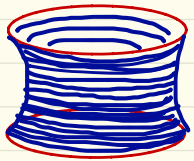

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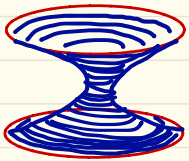
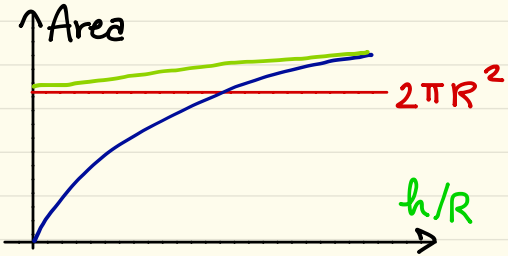


BOTH STABLE

UNSTABLE



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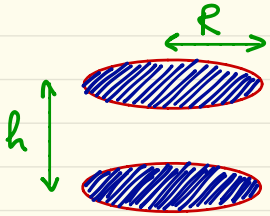

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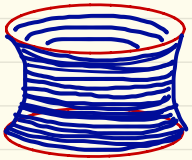

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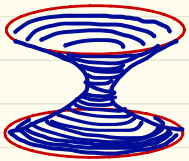
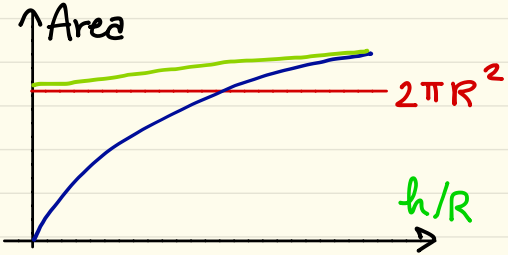
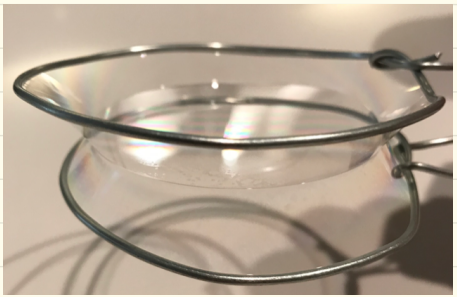


A PRIORI SMOOTHNESS

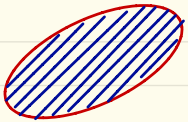




NO LENGHT SCALE

$H_M = 0$

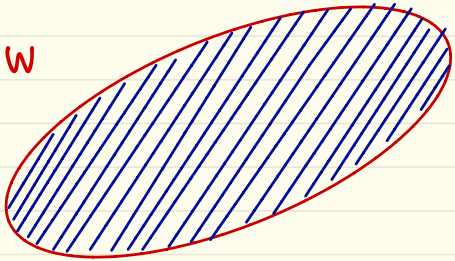
$\partial M = W$



RESCALE $\lambda \gg 1$

$H_{\lambda M} = 0$

$\partial(\lambda M) = \lambda W$



MISSING:

LIMITATION ON SIZE OF ACTUAL SOAP FILMS

NO LENGHT SCALE

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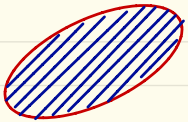

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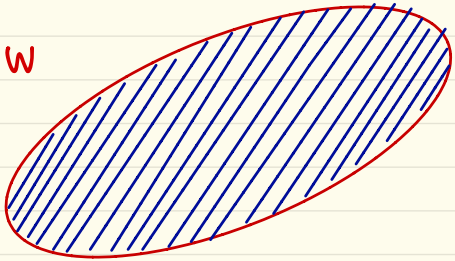


MISSING:

LIMITATION ON SIZE OF ACTUAL SOAP FILMS

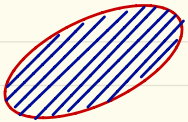
KEY POINT SOAP FILMS ARE ACTUALLY 3D OBJECTS

WITH THICKNESS OF ORDER $\approx 1\,\mu m = 10^{-6}\,m$

NO LENGHT SCALE

$H_M = 0$

$\partial M = W$



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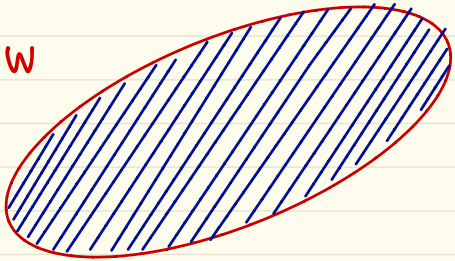

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LIMITATION ON SIZE OF ACTUAL SOAP FILMS

KEY POINT SOAP FILMS ARE ACTUALLY 3D OBJECTS

WITH THICKNESS OF ORDER $\approx 1\,\mu m = 10^{-6}\,m$

CORRECT EQNS BALANCE OF PRESSURES WITH THICKNESS

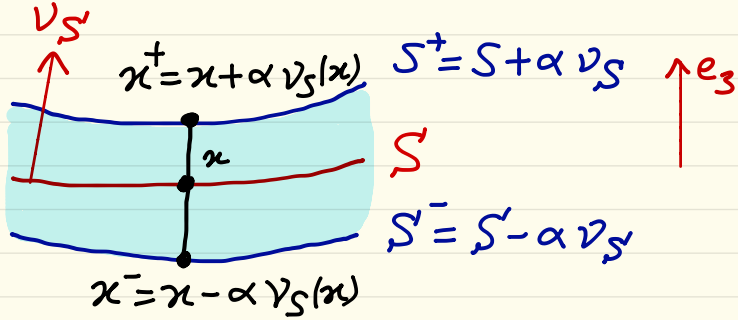
(MORE ACCURATE THAN LAPLACE'S BALANCE!)

DEFAY & PRIGOGINE SURFACE TENSION & ADSORPTION 1966

BALANCE OF PRESSURES

S MIDSURFACE

$\alpha(x)$ THICKNESS



BALANCE OF PRESSURES

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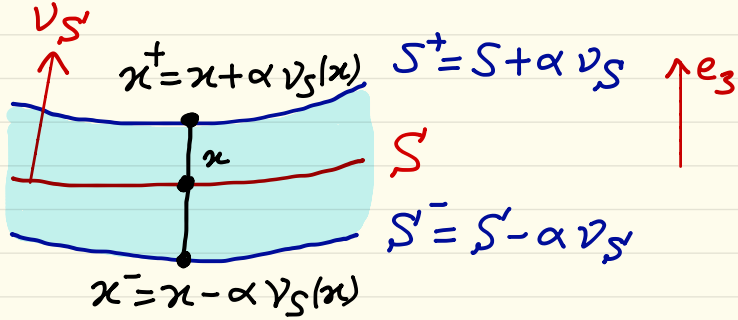

$$p(x^+) = p_0 - \sigma\, H_{S^+}(x^+)$$

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σ SURFACE TENSION N/m

p_0 ATM. PRESSURE N/m^2

BALANCE OF PRESSURES

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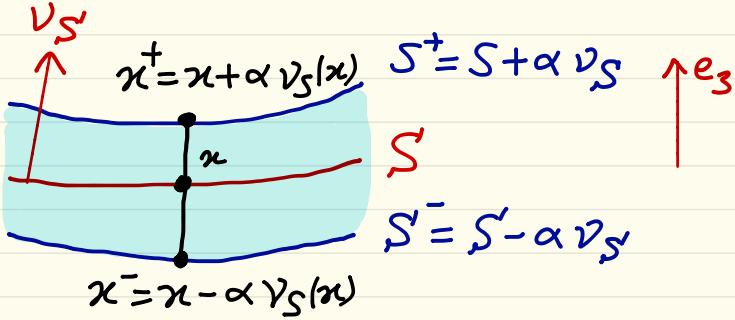

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σ SURFACE TENSION N/m

g EARTH'S GRAVITY m/s^2

ρ VOLUME DENSITY kg/m^3

p_0 ATM. PRESSURE N/m^2

BALANCE OF PRESSURES

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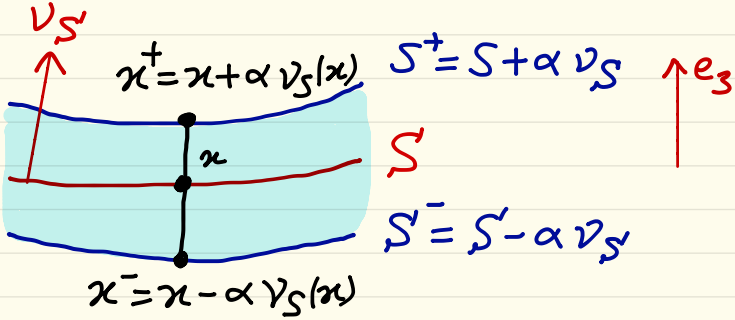

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\end{cases}
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$$= \rho g\, 2\alpha\, (\nu_S \cdot e_3)$$

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BALANCE OF PRESSURES

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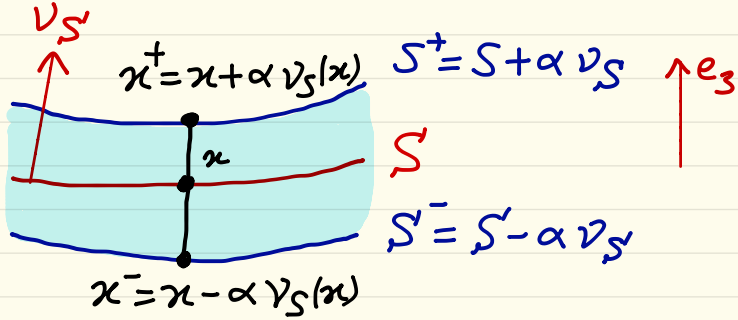

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$$2 \rho g\, \alpha\, \nu_S \cdot e_3 = \sigma \left\{ H_{S^-}(x^-) - H_{S^+}(x^+) \right\}$$

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BALANCE OF PRESSURES

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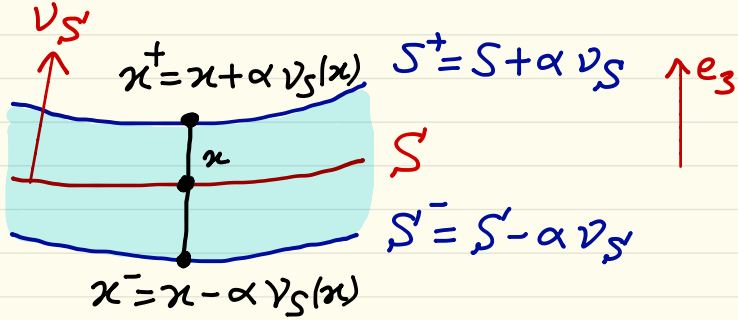

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$$2 \rho g\, \alpha\, \nu_S \cdot e_3$$
$$= \sigma \left\{ H_{S^-}(x^-) - H_{S^+}(x^+) \right\}$$
$$= \sigma \left\{ \sum \frac{k_i}{1 - \alpha k_i} + \sum \frac{k_i}{1 + \alpha k_i} \right\}$$

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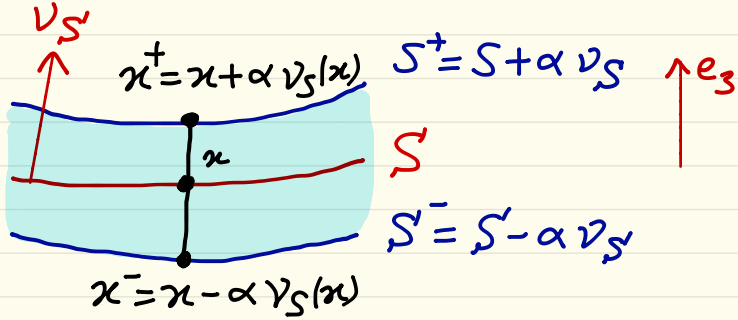

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$$= 2\sigma H_S + O(\alpha^2)$$

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BALANCE OF PRESSURES

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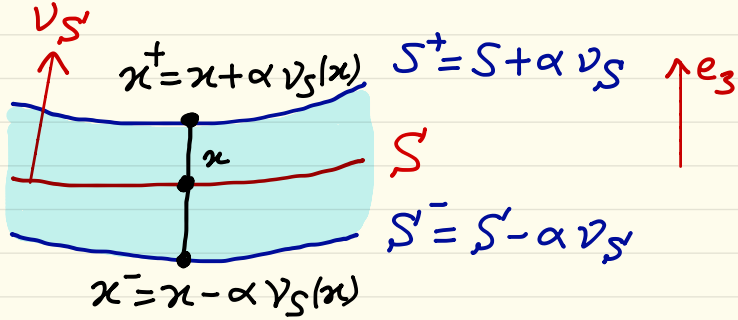

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$$= 2\sigma H_S + O(\alpha^2)$$

$$\boxed{H_S = k^2 \alpha\, (\nu_S \cdot e_3) + O(\alpha^2)}$$

$$k^{-1} = \sqrt{\frac{\sigma}{\rho g}}$$ CAPILLARITY LENGHT

BALANCE OF PRESSURES

$$\boxed{\begin{aligned} p(x^+) &= p_0 - \sigma H_S^+(x^+) \\ p(x^-) &= p_0 + \sigma H_S^-(x^-) \\ p(x^-) - p(x^+) &= \rho g \, 2\alpha \, (\nu_S \cdot e_3) \end{aligned}}$$

MIDSURFACE EQN.

$$\boxed{H_S = k^2 \alpha \, \nu_S \cdot e_3 + O(\alpha^2)}$$

NEED CLARIFICATIONS

BECAUSE...

BALANCE OF PRESSURES

$$p(x^+) = p_0 - \sigma H_S^+(x^+)$$

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BECAUSE...

1) VOLUME MORE FUNDAMENTAL THAN THICKNESS

BALANCE OF PRESSURES

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$$p(x^-) = p_0 + \sigma H_S^-(x^-)$$

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MIDSURFACE EQN.

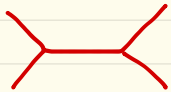
$$H_S = \kappa^2 \alpha \, \nu_S \cdot e_3 + O(\alpha^2)$$

NEED CLARIFICATIONS

BECAUSE...

1) VOLUME MORE FUNDAMENTAL THAN THICKNESS

2) NO MID-SURFACE S FOR SINGULAR CATENOIDS



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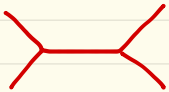

3) WHAT IS THE VARIATIONAL FRAMEWORK?

BALANCE OF PRESSURES

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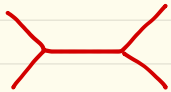


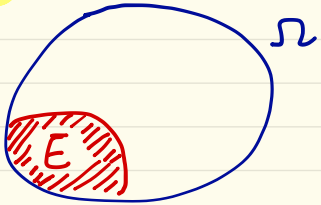
3) WHAT IS THE VARIATIONAL FRAMEWORK?

MECHANICAL EQUILIBRIUM OF 3D REGIONS OF LIQUID

$\Longrightarrow$ CAPILLARITY THEORY: PERIMETER MINIMIZATION

WITH VOLUME CONSTRAINT

CAPILLARITY THEORY



Ω CONTAINER $E \subseteq \Omega$ LIQUID

$\varepsilon = |E|$ VOLUME OF E SMALL

CAPILLARITY ENERGY

$$\sigma \left\{ \mathrm{Area}(E \cap \partial\Omega) + \beta\, \mathrm{Area}(\partial\Omega \cap \partial E) \right\} + \rho g \int_E x_3 \, dx$$

CAPILLARITY THEORY

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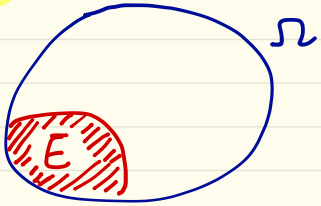

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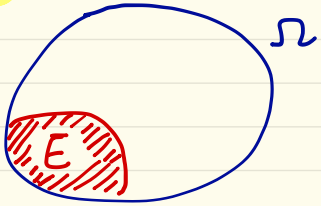
SOAP FILM Ω COMPLEMENT OF WIRE FRAME $W = I_\delta(\Gamma)$

SO $\Omega = \mathbb{R}^3 \setminus W$ UNBOUNDED

CAPILLARITY THEORY



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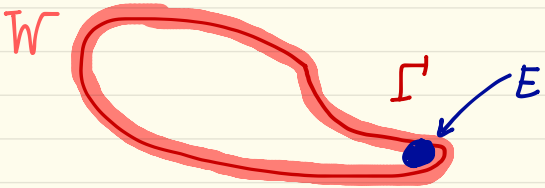
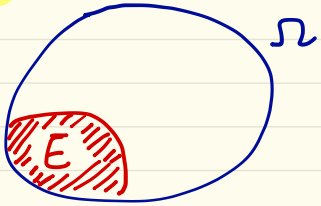


$\Longrightarrow$ WE JUST FIND A SMALL DROPLET NEAR THE POINT OF HIGHEST CURVATURE OF W

BAYLE & ROSALES 2005
FALL 2010
MAGGI & MIHAILA 2015

CAPILLARITY THEORY



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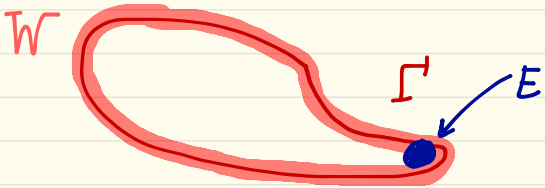


$\Longrightarrow$ WE JUST FIND A SMALL DROPLET NEAR THE POINT OF HIGHEST CURVATURE OF W

$\boxed{\text{WE NEED A SPANNING CONDITION!}}$

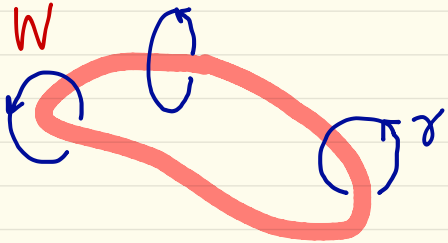
HARRISON-PUGH FORMULATION OF PLATEAU'S PB.

W COMPACT SET IN $\mathbb{R}^{n+1}$

$\mathcal{C}$ FAMILY OF EMBEDDINGS OF S^1

INTO $\Omega = \mathbb{R}^{n+1} \setminus W$ CLOSED BY

HOMOTOPIES RELATIVE TO Ω



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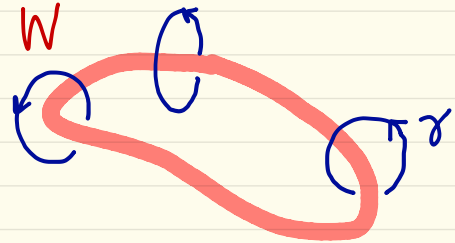
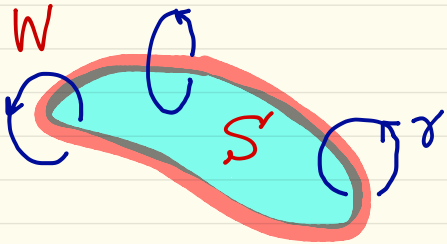


$$\ell = \inf\left\{ \mathcal{H}^n(S) : S \text{ CLOSED IN } \Omega \right.$$
$$\left. S \cap \gamma \neq \emptyset \quad \forall \gamma \in \mathcal{C} \right\}$$



HOMOTOPIC SPANNING CONDITION

EXISTENCE OF MINIMIZERS

HARRISON & PUGH 2013

DE LELLIS & GHIRALDIN & MAGGI 2014

HARRISON-PUGH FORMULATION OF PLATEAU'S PB.

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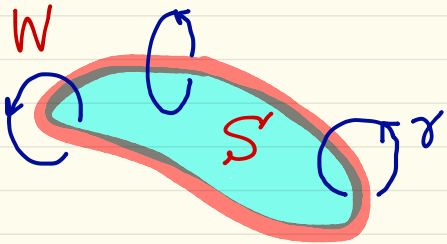

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MINIMALITY W.R.T. LIPSCHITZ DEFORMATIONS

$$\mathcal{H}^n(S) \leq \mathcal{H}^n(\varphi(S))$$

$$\forall\ \varphi \in \mathrm{Lip},\ \{\varphi \neq \mathrm{id}\} \subset\subset \Omega$$

HARRISON-PUGH FORMULATION OF PLATEAU'S PB.

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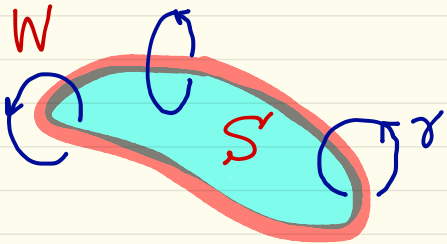

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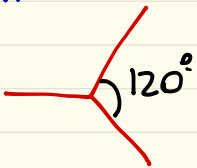
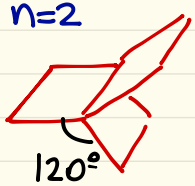
ALMGREN 76

$n=1$



$n=2$



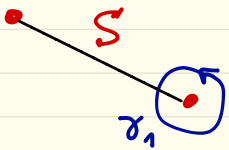
TAYLOR 76 $\Longrightarrow$ PLATEAU'S LAWS

HARRISON-PUGH FORMULATION OF PLATEAU'S PB.

PLANAR EXAMPLES

1 $W=\{p_1,p_2\}$

$\mathcal{C}=\{[\gamma_1]\}$



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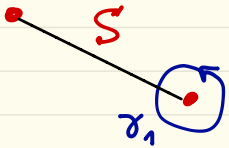

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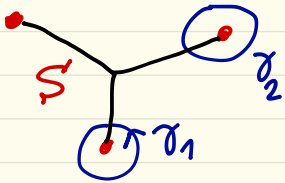

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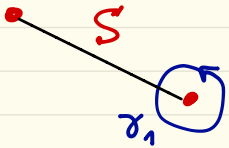

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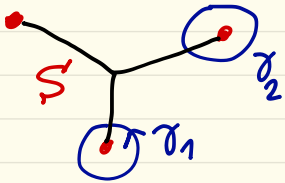

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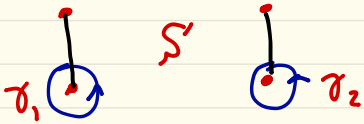

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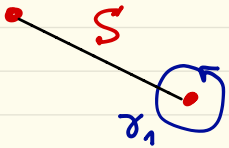

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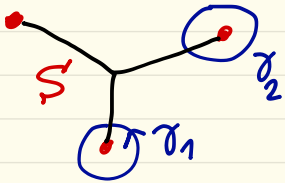

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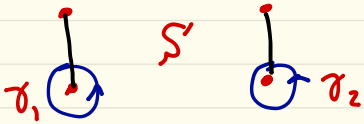

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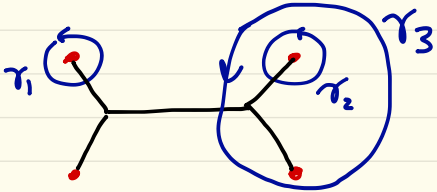

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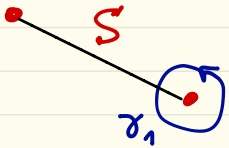

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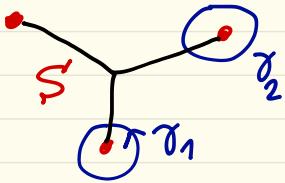

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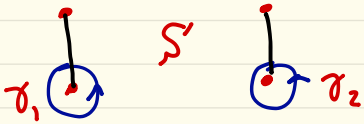

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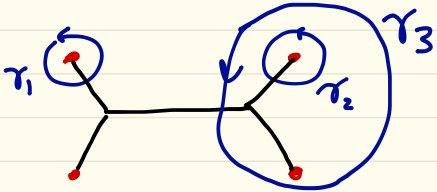
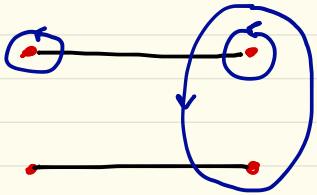


NON UNIQUENESS



CAPILLARITY APPROXIMATION OF PLATEAU PROBLEM

W COMPACT IN $\mathbb{R}^{n+1}$; $\mathcal{C}$ SPANNING CLASS IN $\Omega = \mathbb{R}^{n+1} \setminus W$

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CAPILLARITY WITH $\sigma = 1$, $\beta = 0$, $g = 0$ & HOMOTOPIC SPANNING

$$\psi(\varepsilon) = \inf\{\mathcal{H}^n(\Omega \cap \partial E) : E \subseteq \Omega \text{ OPEN},\ |E| = \varepsilon,\ \Omega \cap \partial E \cap \gamma \neq \emptyset \ \forall \gamma \in \mathcal{C}\}$$

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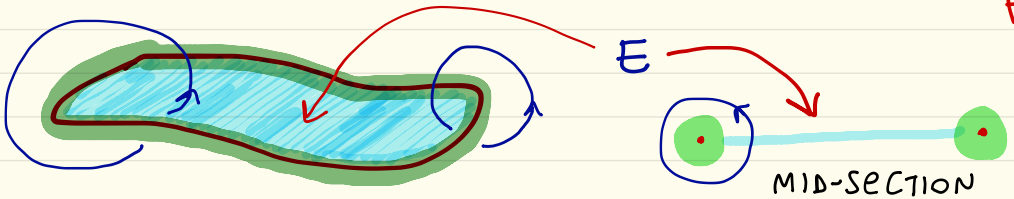

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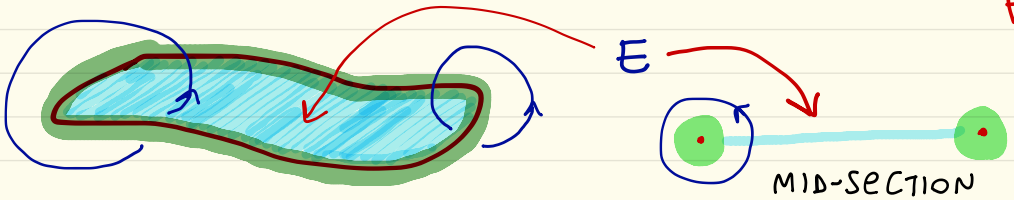

POSSIBLE COLLAPSING OF MINIMIZING SEQUENCES

$E_j \to E$ IN $L^1(\Omega)$

THM ($\exists$ OF GENERALIZED MINIMIZERS)

IF $\ell < \infty$, ∂W SMOOTH, $\exists \tau_0 > 0$ S.T. $\mathbb{R}^{n+1} \setminus I_\tau(W)$ CONNECTED $\forall \tau < \tau_0$

& $\exists \eta_0 > 0$ $\exists S$ MIN. OF ℓ S.T. $\gamma \setminus I_{\eta_0}(S) \neq \emptyset$ $\forall \gamma \in \mathcal{C}$

THEN

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THEN $\forall$ MINIM. SEQUENCE $\{E_j\}$ OF $\psi(\varepsilon)$

$\exists$ K COMPACT IN Ω, $\exists$ $E \subseteq \Omega$, $|E| = \varepsilon$, $\Omega \cap \partial E \subseteq K$

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S.T. 1 $K \cap \gamma \neq \emptyset$ $\forall \gamma \in \mathcal{C}$ 2 $E_j \to E$ $L^1(\mathbb{R}^{n+1})$

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3 $\mathcal{H}^n L(\Omega \cap \partial E_j) \xrightarrow{*} 2\mathcal{H}^n L(K \setminus \partial^* E) + \mathcal{H}^n L(\Omega \cap \partial^* E)$

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④ $\psi(\varepsilon) = 2\mathcal{H}^n(K \setminus \partial^* E) + \mathcal{H}^n(\Omega \cap \partial^* E)$

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S.T. ① $K \cap \gamma \neq \emptyset$ $\forall \gamma \in \mathcal{C}$ ② $E_j \to E$ $L^1(\mathbb{R}^{n+1})$

③ $\mathcal{H}^n \llcorner (\Omega \cap \partial E_j) \xrightarrow{*} 2\mathcal{H}^n \llcorner (K \setminus \partial^* E) + \mathcal{H}^n \llcorner (\Omega \cap \partial^* E)$

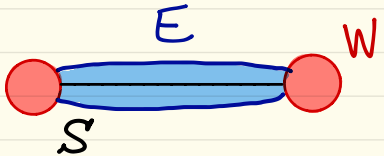
④ $\psi(\varepsilon) = 2\mathcal{H}^n(K \setminus \partial^* E) + \mathcal{H}^n(\Omega \cap \partial^* E)$

STANDARD MINIMIZER IF $K = \Omega \cap \partial E$

COLLAPSED MINIMIZER IF $\mathcal{H}^n(K \setminus \partial E) > 0$

PLANAR EXAMPLES

1 W TWO DISKS



S SEGMENT S.T. $\mathcal{H}^1(S) = \ell$

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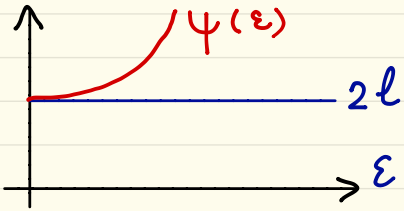
TWO CIRCULAR ARCS
OF CURVATURE $O(\varepsilon)$

$$\psi(\varepsilon) = 2\ell + C\varepsilon^2 + o(\varepsilon^2)$$



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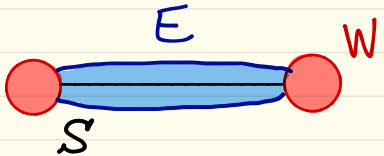

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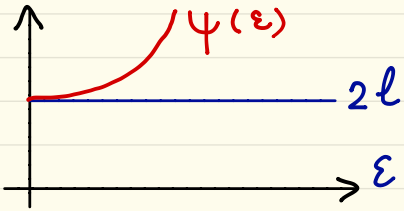
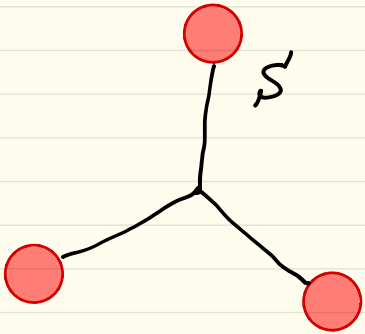


2 W THREE DISKS



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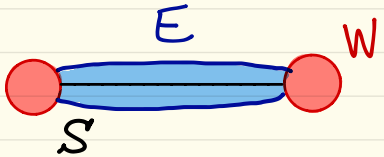

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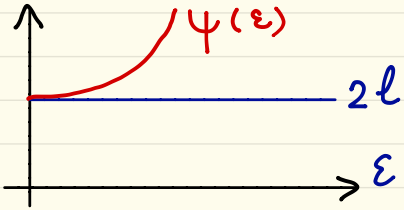

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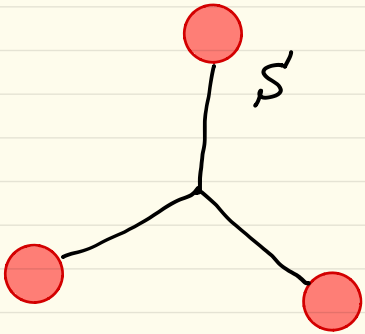

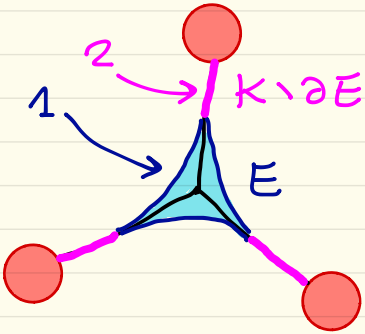

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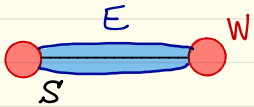

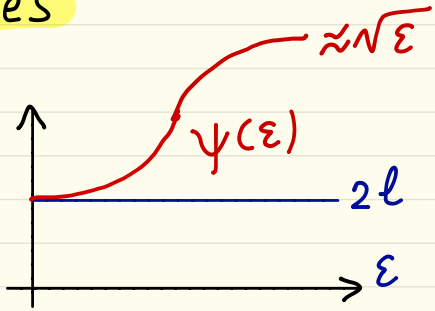

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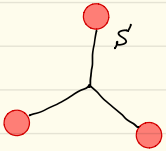
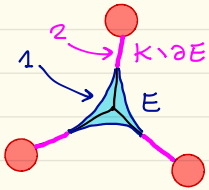
S 3 SEGMENTS





(K,E) GENERALIZED MW.

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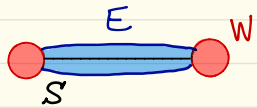

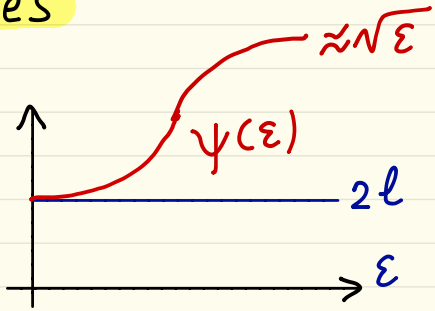

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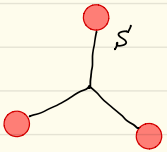

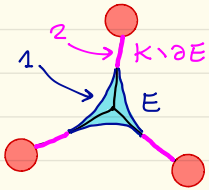

(K,E) GENERALIZED MW.

$K\setminus\partial E$ COLLAPSED PART
TWICE THE LENGHT

∂E 3 CIRCULAR ARCS OF
CURVATURE $O(-1/\sqrt{\varepsilon})$

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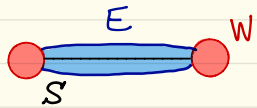

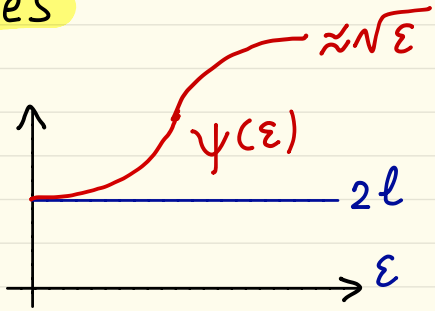

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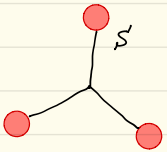

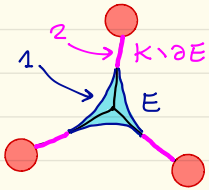

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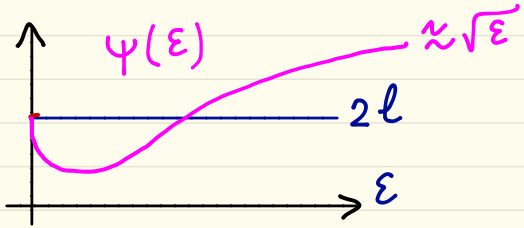

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THM CONVERGENCE TO PLATEAU

UNDER THE SAME ASSUMPTIONS OF THE $\exists$ THM:

1. $\psi(\varepsilon)$ IS LOWERSEMICONTINUOUS WITH $\psi(\varepsilon) \leq 2\ell + C\varepsilon^{n/n+1}$
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   THEN $\exists$ S MIN OF ℓ S.T.

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CONJECTURE NO COLLAPSING IF EVERY MIN OF ℓ IS SMOOTH

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CONJECTURE NO COLLAPSING IF EVERY MIN OF ℓ IS SMOOTH

SELECTION MORE CURVATURE, MORE SINGULARITIES

THM BASIC REGULARITY

IF (K,E) GEN MIN OF $\psi(\varepsilon)$, $\mathcal{F}(K,E) = 2\mathcal{H}^n(K \setminus \partial^* E) + \mathcal{H}^n(\Omega \cap \partial^* E)$

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THEN $\mathcal{F}(K,E) \leq \mathcal{F}(\varphi(K), \varphi(E))$, $\forall \varphi : \Omega \to \Omega$ DIFFEOMORPHISM

S.T. $|E| = |\varphi(E)|$

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IN PARTICULAR:

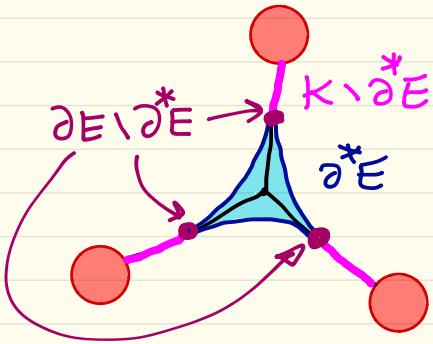
$\exists \lambda \in \mathbb{R}$, $\exists \Sigma \subseteq K$ CLOSED S.T.

$K \setminus (\Sigma \cup \partial E)$ C^∞ MINIMAL SURFACE

$\Omega \cap \partial^* E$ C^∞, CONSTANT MEAN CURV. λ

$\mathcal{H}^n(\Sigma \setminus \partial E) = 0$;

$\partial E \setminus \partial^* E$ HAS EMPTY INTERIOR



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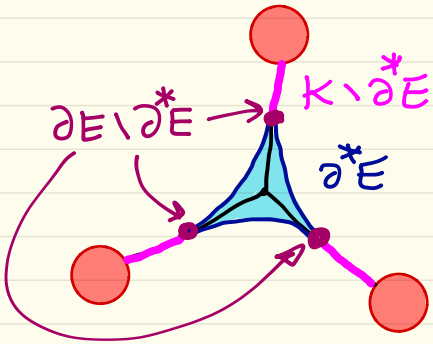

STRONGER REGULARITY EXPECTED e.g. WORK IN PROGRESS

INDICATES THAT $\Sigma \cap (K \setminus \partial E) = \emptyset$ IF $n = 1, 2$

THM COLLAPSING & CONVEX HULL PROPERTY

UNDER THE SAME ASSUMPTIONS OF THE $\exists$ THM:

IF (K,E) GEN MIN OF $\psi(\varepsilon)$ WITH E.L. MULTIPLIER λ

THEN 1 IF $K \setminus \overline{E} \neq \emptyset$, THEN $\lambda \leq 0$

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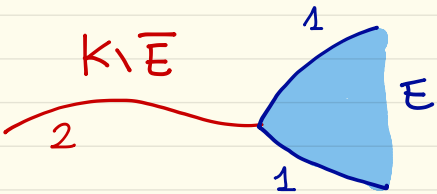
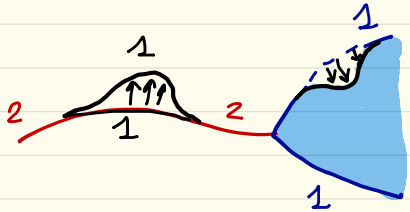
RMK PROOF OF 1 NEEDS NON-MAPPING VARIATIONS





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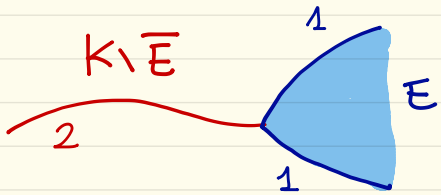

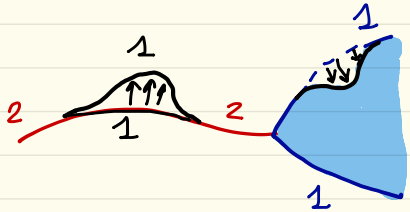

AREA INCREASE QUADRATIC IN VOLUME ADDED v (BY $H_{K \setminus \partial E} = 0$)

FOLLOWED BY AREA VARIATION λv TO RESTORE TOT. VOLUME

BIBLIOGRAPHY

MAGGI, SCARDICCHIO, STUVARD DCDS 2019 CAFFARELLI'S 70TH

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KING, MAGGI, STUVARD ARXIV FEB 2020

COLLAPSING & THE CONVEX HULL PROPERTY IN A SOAP FILM

CAPILLARITY MODEL