THREE LECTURES ON ISOPERIMETRIC & PLATEAU-TYPE PROBLEMS

FRANCESCO MAGGI

UNIVERSITY OF TEXAS AT AUSTIN

GEOMETRIC ANALYSIS & PDE on GARDA LAKE

8-10 JUNE 2022

LECTURE 1

QUANTITATIVE ISOPERIMETRY, IMPROVED CONVERGENCE & ISOPERIMETRIC RESIDUES

QUANTITATIVE ISOPERIMETRY

ISOPERIMETRIC PROBLEM

$$\Psi_{iso}(v) = \inf\{P(E) : |E| = v\} \quad v > 0 \text{ GIVEN}$$

$$|E| = \mathcal{L}^n(E) = \text{VOLUME OF } E$$

$$P(E) = \mathcal{H}^{n-1}(\partial E) = \text{PERIMETER OF } E$$

QUANTITATIVE ISOPERIMETRY

ISOPERIMETRIC PROBLEM $\Psi_{iso}(v) = \inf\{P(E) : |E| = v\}$ $v > 0$ GIVEN

$|E| = \mathcal{L}^n(E)$ = VOLUME OF E

$P(E) = \mathcal{H}^{n-1}(\partial E)$ = PERIMETER OF E

ISOPERIMETRIC INEQUALITY

$P(E) \geq n\omega_n^{1/n} |E|^{\frac{n-1}{n}}$ WITH = IFF $E = B_r(x)$

$\omega_n = |B_1|$

$n\omega_n^{1/n} |E|^{\frac{n-1}{n}}$ = PERIMETER BALL OF VOLUME $|E|$.

$\Rightarrow \Psi_{iso}(v) = n\omega_n^{1/n} v^{\frac{n-1}{n}}$

QUANTITATIVE ISOPERIMETRY

ISOPERIMETRIC PROBLEM $\Psi_{iso}(v) = \inf\{P(E) : |E| = v\} = n\omega_n^{1/n} v^{(n-1)/n}$ $(v > 0)$

ISOPERIMETRIC INEQUALITY $P(E) \geq n\omega_n^{1/n} |E|^{\frac{n-1}{n}}$ WITH $=$ IFF $E = B_r(x)$

QUANTITATIVE ISOP.

IF $P(E) = (1+\delta)\, n\omega_n^{1/n} |E|^{\frac{n-1}{n}}$, THEN,

HOW FAR IS E FROM $B_r(x)$ IN TERMS OF δ ?

QUANTITATIVE ISOPERIMETRY

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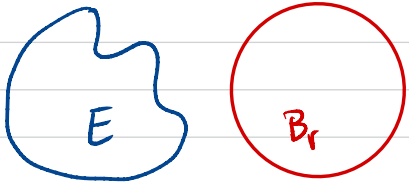
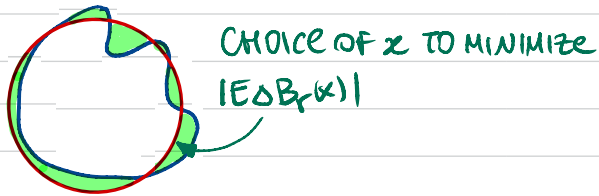
HOW FAR IS E FROM $B_r(x)$ IN TERMS OF δ ?

THM (FUSCO M. PRATELLI '05) $\delta \geq c(n) \inf_{x \in \mathbb{R}^n} \left(\frac{|E \Delta B_r(x)|}{|E|} \right)^2$ (r S.T. $|B_r| = |E|$).





QUANTITATIVE ISOPERIMETRY

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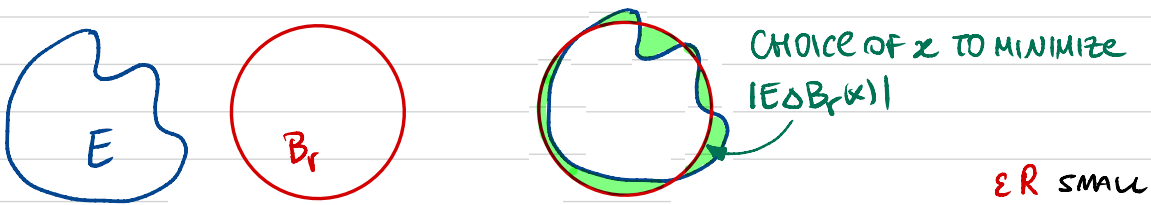
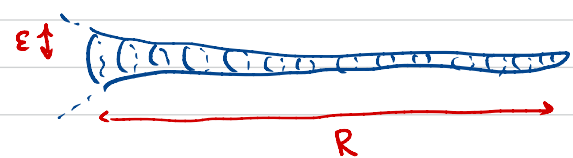


RMK WHEN $n \geq 3$ LARGE DIAMETER WITH SMALL PERIMETER & VOLUME



AN APPLICATION OF QUANT. ISOP.

CAPILLARITY-TYPE ENERGY: DATES BACK TO GAUSS ~ 1820

UNDER A VOLUME CONSTRAINT ON $E \subseteq \mathbb{R}^n$

CONSIDER MINIMIZERS, LOCAL MINIMIZERS & CRITICAL POINTS OF

$$\mathcal{F}_g(E) = \sigma P(E) + \int_E g(x)\,dx$$ = SURFACE TENSION ENERGY + POTENTIAL ENERGY

$\sigma > 0$ SURFACE TENSION CONSTANT

$g : \mathbb{R}^n \to \mathbb{R}$ POTENTIAL ENERGY DENSITY

(FROM THE START, $\sigma = 1$)

E.G. $g(x) = \rho g_0 x_n$ IF $n = 3$

AN APPLICATION OF QUANT. ISOP.

CAPILLARITY-TYPE ENERGY: $\mathcal{F}_g(E) = P(E) + \int_E g(x)\,dx$

ARCHETYPICAL DROPLET PROBLEM $\Psi_g(v) = \inf\{\mathcal{F}_g(E) : |E| = v\}$ $\quad v > 0$ GIVEN

CAPILLARITY REGIME IF v IS SMALL THEN

$P(E) \approx v^{\frac{n-1}{n}}$ DOMINATES $\int_E g \approx v$

& SO $\Psi_g(v) \approx \Psi_{iso}(v)$

IN PARTICULAR WE EXPECT

$E \approx B_r(x)$ IF E MINIMIZER OF $\Psi_g(v)$, v SMALL

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$$E \approx B_r(x) \text{ IF } E \text{ MINIMIZER OF } \Psi_g(v)$$

TEST FOR r S.T. $v = |B_r|$, WE HAVE $P(B_r) = c(n)\, v^{\frac{n-1}{n}}$

$$P(E) + \int_E g \leq P(B_r) + \int_{B_r} g$$

AN APPLICATION OF QUANT. ISOP.

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$$P(E) - P(B_r) \leq \int_{B_r} g - \int_E g$$

$$\frac{P(E)}{P(B_r)} - 1 \leq \frac{1}{P(B_r)}\left\{\int_{B_r} g - \int_E g\right\}$$

AN APPLICATION OF QUANT. ISOP.

CAPILLARITY-TYPE ENERGY: $\mathcal{F}_g(E) = P(E) + \int_E g(x)\,dx$

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$$P(E) - P(B_r) \leq \int_{B_r} g - \int_E g$$

$$\delta_{iso}(E) = \frac{P(E)}{P(B_r)} - 1 \leq \frac{1}{P(B_r)}\left\{\int_{B_r} g - \int_E g\right\} \leq \frac{C(n,g)\, v}{v^{\frac{n-1}{n}}}$$

AN APPLICATION OF QUANT. ISOP.

CAPILLARITY-TYPE ENERGY: $\mathcal{F}_g(E) = P(E) + \int_E g(x)\,dx$

ARCHETYPICAL DROPLET PROBLEM $\Psi_g(v) = \inf\{\mathcal{F}_g(E) : |E| = v\}$ $\quad v > 0$ GIVEN

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$$E \approx B_r(x) \text{ IF } E \text{ MINIMIZER OF } \Psi_g(v)$$

TEST FOR r S.T. $v = |B_r|$, WE HAVE $P(B_r) = c(n)\, v^{\frac{n-1}{n}}$

$$c(n)\left(\inf_{x\in\mathbb{R}^n} \frac{|E\Delta B_r(x)|}{|E|}\right)^2 \le \delta_{iso}(E) \le \frac{C(n,g)\,v}{v^{\frac{n-1}{n}}} = C(n,g)\, v^{1/n}$$

$$\boxed{\inf_{x\in\mathbb{R}^n} \frac{|E\Delta B_r(x)|}{|E|} \le C(n,g)\, v^{1/2n}}$$

A MINIMIZER E OF

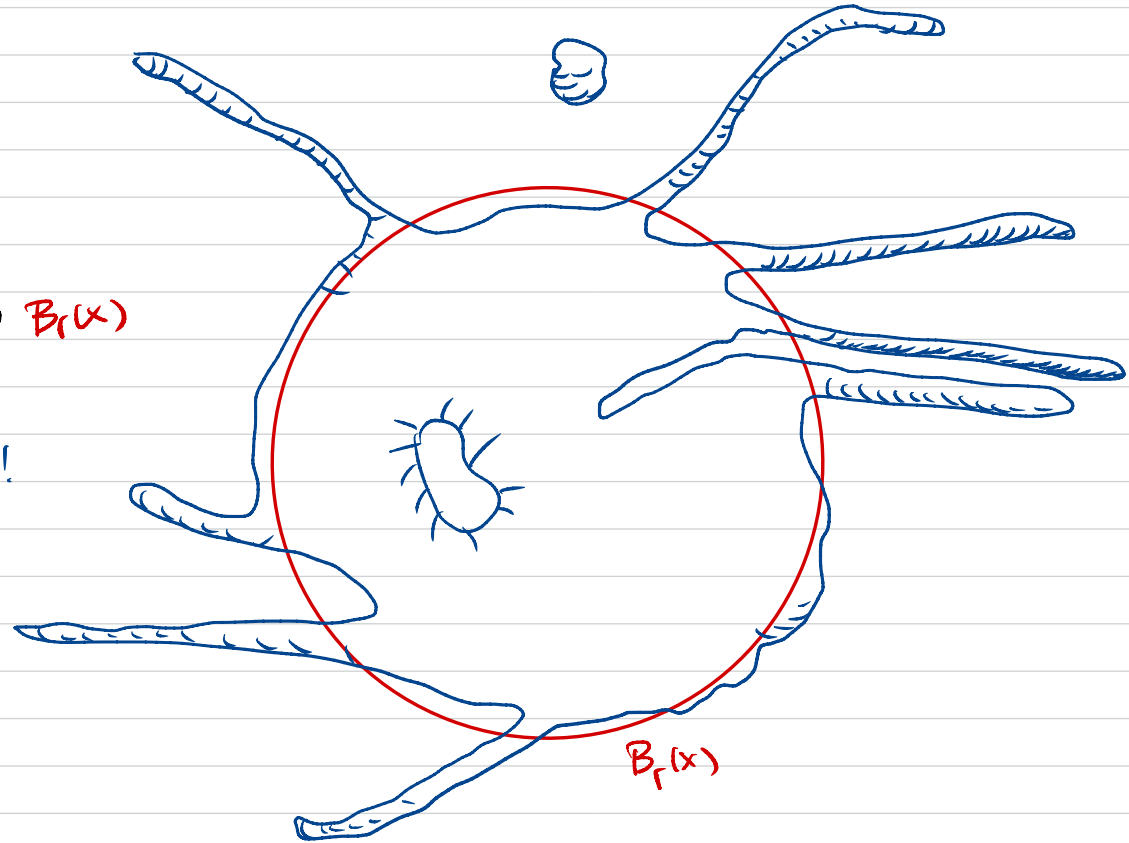
$P(E) + \int_E g$ WITH

$v = |E|$ SMALL IS

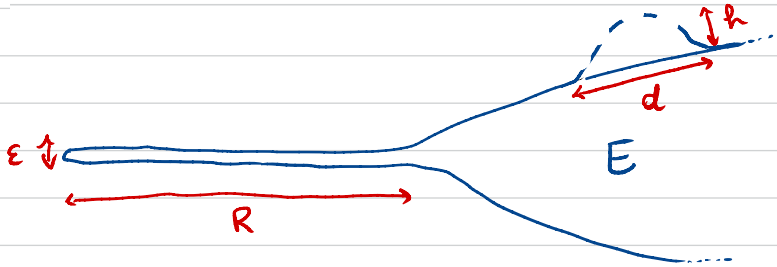
CLOSE IN VOLUME TO $B_r(x)$

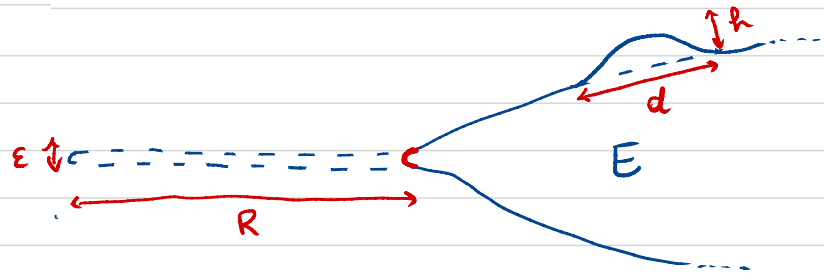
CLEARLY NON-OPTIMAL!

"REPLACING TENTACLES WITH BUMPS"



REPLACING TENTACLES WITH BUMPS IN $\mathbb{R}^3$





$$\Delta \mathrm{Vol} \approx -\pi \varepsilon^2 R + \pi d^2 h \equiv 0 \implies h = \varepsilon^2 R / d^2 \approx \varepsilon^2 R \quad (d \sim 1)$$

$$\Delta \mathrm{Per} \approx -2\pi \varepsilon R + \underbrace{\pi d^2 \sqrt{1 + (h/d)^2} - \pi d^2}_{\leq C d^2 (h/d)^2 = C h^2} \approx -\varepsilon R + \varepsilon^4 R^2$$

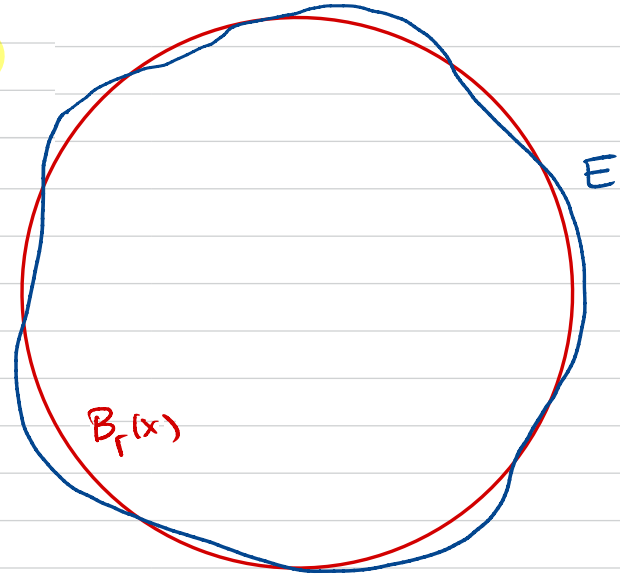
$$\approx \varepsilon R(-1 + \varepsilon^2) < 0.$$

$(\varepsilon R < 1)$

ELABORATING ON THE TENTACLES VS BUMPS IDEA...

A MINIMIZER E OF $P(E) + \int_E g$ WITH $v = |E|$ SMALL IS CLOSE IN VOLUME TO $B_r(x)$ $\boxed{\text{THEREFORE}}$

E IS CLOSE TO $B_r(x)$ IN C^1



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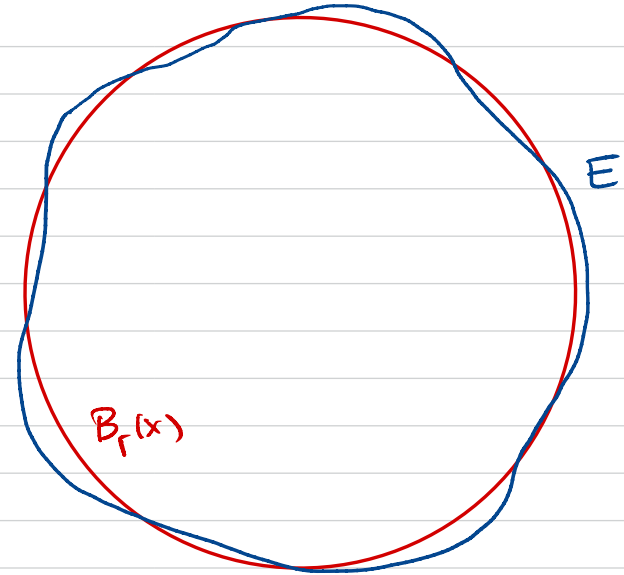
E IS CLOSE TO $B_r(x)$ IN C^1

$\partial E = \{ y + u(y)\, \nu_{B_r(x)}(y) : y \in \partial B_r(x) \}$

$\|u\|_{C^1(\partial B_r(x))}$ SMALL IN TERMS OF v



THM (FIGALLI M '10) C^2-CLOSE & CONVEXITY

NOTATION: ν_F OUTER UNIT NORMAL TO F

$\boxed{\text{EXAMPLE OF IMPROVED CONVERGENCE}}$

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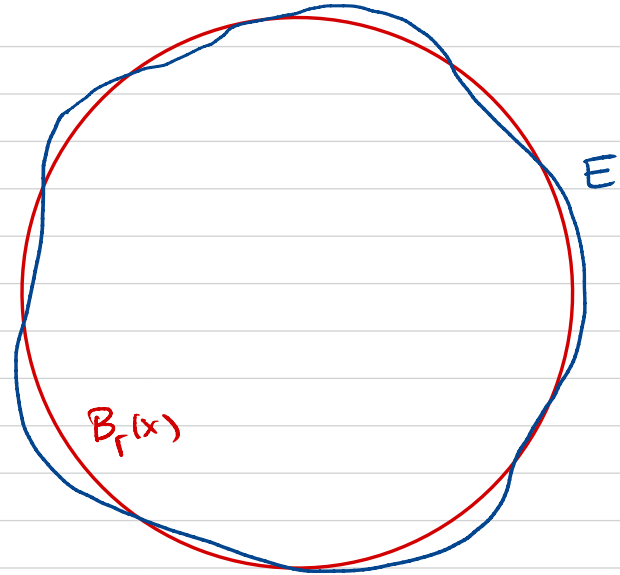

NOTATION: ν_F OUTER UNIT NORMAL TO F

OTHER EXAMPLES: PROOF OF QUANT. ISOP. (CICALESE & LEONARDI '10)

MINIMALITY OF BALLS IN FRACTIONAL GAMOW TYPE PROBS (FIGALLI FUSCO M. MILLOT MORINI '14)

IMPROVED CONVERGENCE

AN OLD IDEA IN GEOMETRIC MEASURE THEORY

OLDER PAPER I KNOW OF: MIRANDA 1967

IMPROVED CONVERGENCE A MODEL STATEMENT:

IF $\{M_j\}_j, M$ (SMOOTH) HYPERSURFACES IN $\Omega \subseteq \mathbb{R}^n$ OPEN

HP1 UNIFORM MEAN CURVATURE BOUND $|H_{M_j}| \leq \Lambda$ ON Ω

HP2 M_j "WEAKLY CONVERGE" TO M SO THAT

$$\mathcal{H}^{n-1}(M_j \cap B_\rho(x)) \to \mathcal{H}^{n-1}(M \cap B_\rho(x)) \quad \forall x \in \Omega \ \& \ \text{a.e. } \rho < \operatorname{dist}(x, \partial\Omega)$$

RMK CONVERGENCE IN HP2 COMPATIBLE WITH TENTACLES

BUT PREVENTS SURFACE STACKING!

IMPROVED CONVERGENCE A MODEL STATEMENT:

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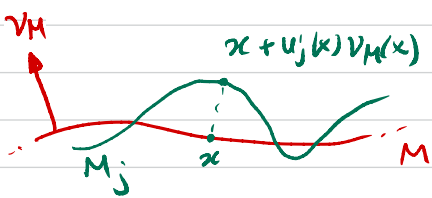
HP2 $\mathcal{H}^{n-1}(M_j \cap B_\rho(x)) \to \mathcal{H}^{n-1}(M \cap B_\rho(x))$ $\forall x \in \Omega$ & a.e. $\rho < \mathrm{dist}(x, \partial\Omega)$

THEN

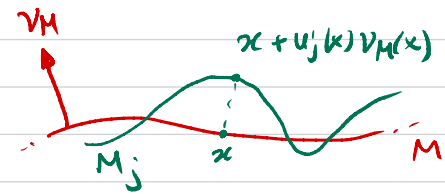
$\forall \Omega' \subset\subset \Omega$, $j \geq j_{\Omega'}$, WE HAVE

$$M_j \cap \Omega' = \{x + u_j(x)\nu_M(x) : x \in M\} \cap \Omega', \quad \|u_j\|_{C^1(\Omega')} \to 0 \quad j \to \infty.$$



THAT IS "$M_j \to M$ IN $C^1_{loc}(\Omega)$"

IMPROVED CONVERGENCE A MODEL STATEMENT:



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HP2 $\mathcal{H}^{n-1}(M_j \cap B_\rho(x)) \to \mathcal{H}^{n-1}(M \cap B_\rho(x))$ $\forall x \in \Omega$ & a.e. $\rho < \mathrm{dist}(x, \partial\Omega)$

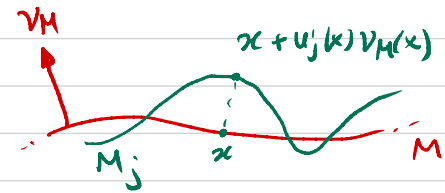
THEN "$M_j \longrightarrow M$ IN $C^1_{loc}(\Omega)$"

IN OUR EXAMPLE

$$\Omega = \mathbb{R}^n, \qquad M_j = \partial\left[\frac{E_{v_j}}{v_j^{1/n}}\right] \ (v_j \to 0^+), \qquad M = \partial B_{r_0} \quad (|B_{r_0}| = 1)$$

E_{v_j} MINIMIZER OF $\Psi_g(v_j)$

IMPROVED CONVERGENCE A MODEL STATEMENT:



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THEN "$M_j \to M$ IN $C^1_{loc}(\Omega)$"

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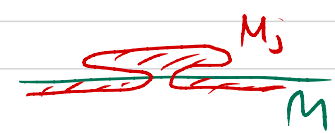
HP1 FOLLOWS FROM EULER-LAGRANGE EQN $H_{\partial E_j}(x) + g(x) = C_0 v_j^{-1/n}$ $\forall x \in \partial E_j$

HP2 FOLLOWS BY $[E_{v_j}/v_j^{1/n}] \to B_{r_0}$ IN $L^1(\mathbb{R}^n)$ ($\Leftarrow$ QUANT. ISOPERIMETRY)

PLUS MINIMALITY OF E_{v_j} IN $\Psi_g(v_j)$



IMPROVED CONVERGENCE STATEMENTS DESCENDS FROM

FLATNESS CRITERIA (A.K.A. ε-REGULARITY THMS)

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I.C. STATEMENT $\Leftarrow$ FLATNESS CRITERION

MIRANDA'S 1967 I.C. $\Leftarrow$ DE GIORGI'S REGULARITY FOR LOCAL PERIM. MINIMIZERS (1960)

"MODEL STATEMENT" FROM PREVIOUS SLIDES $\Leftarrow$ ALLARD'S REGULARITY THEOREM (1968)

FLATNESS CRITERION (ALLARD'S REGULARITY THM)

IF M (SMOOTH) HYPERSURFACE IN $\Omega \subseteq \mathbb{R}^n$ OPEN, $|H_M| \leq \Lambda$ ON Ω

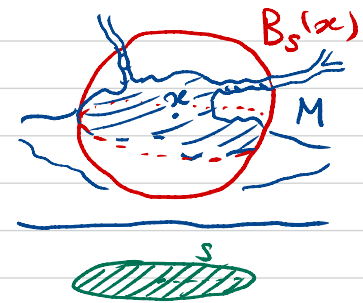
& $s < \varepsilon_0/\Lambda$ SUCH THAT $\boxed{\mathcal{H}^{n-1}(B_s(x) \cap M) \leq (1+\varepsilon_0)\,\omega_n s^{n-1}}$ $(x \in M)$

THEN THERE IS H HYPERPLANE, $u : H \to \mathbb{R}$, $(H = T_x M)$

SUCH THAT $\begin{cases} M \cap B_{\varepsilon_0 s}(x) = \{x + u(x)\nu_H ; x \in H\} \cap B_{\varepsilon_0 s}(x) \\ \|u\|_{C^1} < \sigma \end{cases}$ $\varepsilon_0 = \varepsilon_0(n, \Lambda, \sigma)$

$\boxed{\mathcal{H}^{n-1}(B_s(x) \cap M) \leq (1+\varepsilon_0)\,\omega_n s^{n-1}}$:

I.E. $\dfrac{\text{AREA OF } M \text{ INSIDE } B_s(x)}{\text{AREA OF } (n-1) \text{ DIM DISK RADIUS } s}$ =



NOT TOO MUCH LARGER THAN 1

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REMARK IT IS OBVIOUS BY SMOOTHNESS OF M THAT $\exists\, s_x > 0$ (UNIF. BND FROM BELOW IF $x \in \Omega'$, $\Omega' \subset\subset \Omega$ FIXED) S.T.

$$\begin{cases} M \cap B_{s_x}(x) = \{x + u(x)\nu_H ; x \in H\} \cap B_{s_x}(x) \\ \|u\|_{C^1} < \sigma \end{cases}$$

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THE POINT HERE IS RELATING s_x TO Λ & TO THE CONTROL QUANTITY

$$\Theta(s) = \frac{\mathcal{H}^{n-1}(B_s(x) \cap M)}{\omega_n s^{n-1}}$$

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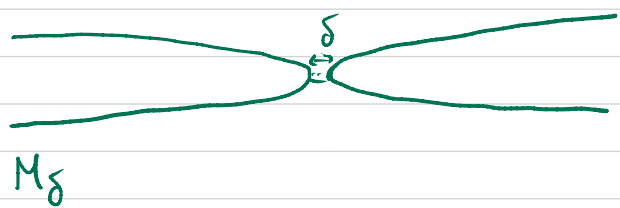
& $s < \varepsilon_0/\Lambda$ SUCH THAT $\mathcal{H}^{n-1}(B_s(x) \cap M) \leq (1+\varepsilon_0)\,\omega_n s^{n-1}$ $(x \in M)$

THEN THERE IS H HYPERPLANE, $u : H \to \mathbb{R}$, $\|u\|_{C^1} < \sigma$

SUCH THAT $M \cap B_{\varepsilon_0 s}(x) = \{x + u(x)\nu_H ; x \in H\} \cap B_{\varepsilon_0 s}(x)$ $\qquad \varepsilon_0 = \varepsilon_0(n, \Lambda, \sigma)$

REMARK: ONE MUST RESORT TO A QUANTITY LIKE $\theta(s) = \dfrac{\mathcal{H}^{n-1}(M \cap B_s(x))}{\omega_n s^{n-1}}$! CANNOT JUST USE Λ!

EX: CATENOIDS WITH WAIST δ $[\, H_{M_\delta} = 0 \ \forall \delta \,]$



"PROOF" OF IMPROVED CONVERGENCE

IF $\{M_j\}_j$, M (SMOOTH) HYPERSURFACES IN $\Omega \subseteq \mathbb{R}^n$ OPEN

HP1 UNIFORM MEAN CURVATURE BOUND $|H_{M_j}| \leq \Lambda$ ON Ω

HP2 $\mathcal{H}^{n-1}(M_j \cap B_\rho(x)) \to \mathcal{H}^{n-1}(M \cap B_\rho(x))$ $\forall x \in \Omega$ & a.e. $\rho < \operatorname{dist}(x, \partial\Omega)$

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$$M_j \cap \Omega' = \{x + u_j(x)\nu_M(x) : x \in M\} \cap \Omega', \quad \|u_j\|_{C^1(\Omega')} \to 0 \quad j \to \infty.$$

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INDEED M SMOOTH $\Rightarrow$ $\exists\, s_0 > 0$ S.T. $\mathcal{H}^{n-1}(M \cap B_{s_0}(x)) \leq (1 + \varepsilon_0/2)\,\omega_n s_0^{n-1}$, $\forall x \in M \cap \Omega'$

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BY HP1 WE CAN APPLY ALLARD TO EACH M_j & TO M

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BY HP1 WE CAN APPLY ALLARD TO EACH M_j & TO M $\Rightarrow \exists\, H_x$ HYPERPLANE S.T.

M_j, M GRAPHS OF $f_j, f : H_x \to \mathbb{R}$ ON $B_{\varepsilon_0 s_0}(x)$ & $f_j \to f$ IN C^1 BY HP2

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M_j, M GRAPHS OF $f_j, f : H_x \to \mathbb{R}$ ON $B_{\varepsilon_0 s_0}(x)$ & $f_j \to f$ IN C^1 BY HP2

CONCLUDE BY COVERING & REPARAMETRIZATION OF M_j OVER M

A MESOSCALE FLATNESS CRITERION & EXTERIOR ISOP. (M-NOVACK '22)

DATA: W COMPACT SET $\subseteq \mathbb{R}^{n+1}$ $\quad v>0$ VOLUME CONSTRAINT

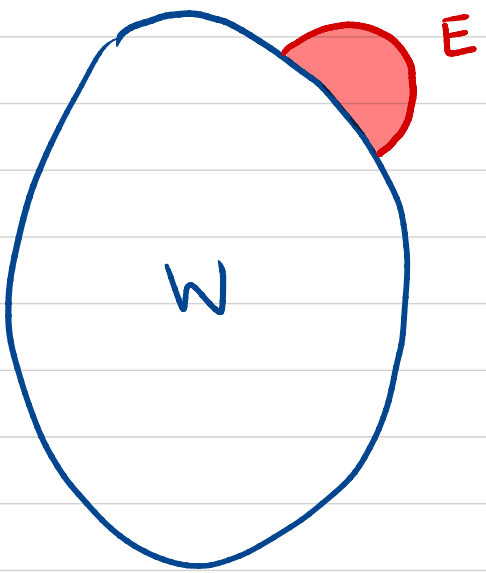
PROBLEM: $\gamma_W(v) = \inf\{ P(E;\Omega) : E \subseteq \Omega = \mathbb{R}^{n+1} \setminus W \,;\, |E| = v \}$

$P(E;\Omega) = \mathcal{H}^n(\Omega \cap \partial E)$ PERIMETER OF E RELATIVE TO Ω

BASICS: MINIMIZERS E_v EXIST $\forall v>0$

$\Omega \cap \partial E_v$ HAS CONSTANT MEAN CURV. H_{E_v}

$\nu_{E_v} \cdot \nu_W = 0$ IF $\partial W \in C^{1,1}$ (YOUNG'S LAW)



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$|E| = v \gg 1$

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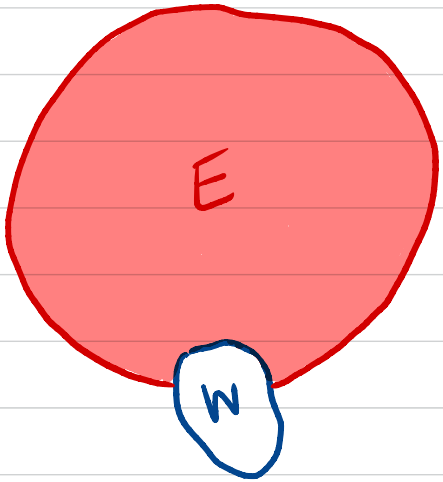


LARGE VOLUME REGIME

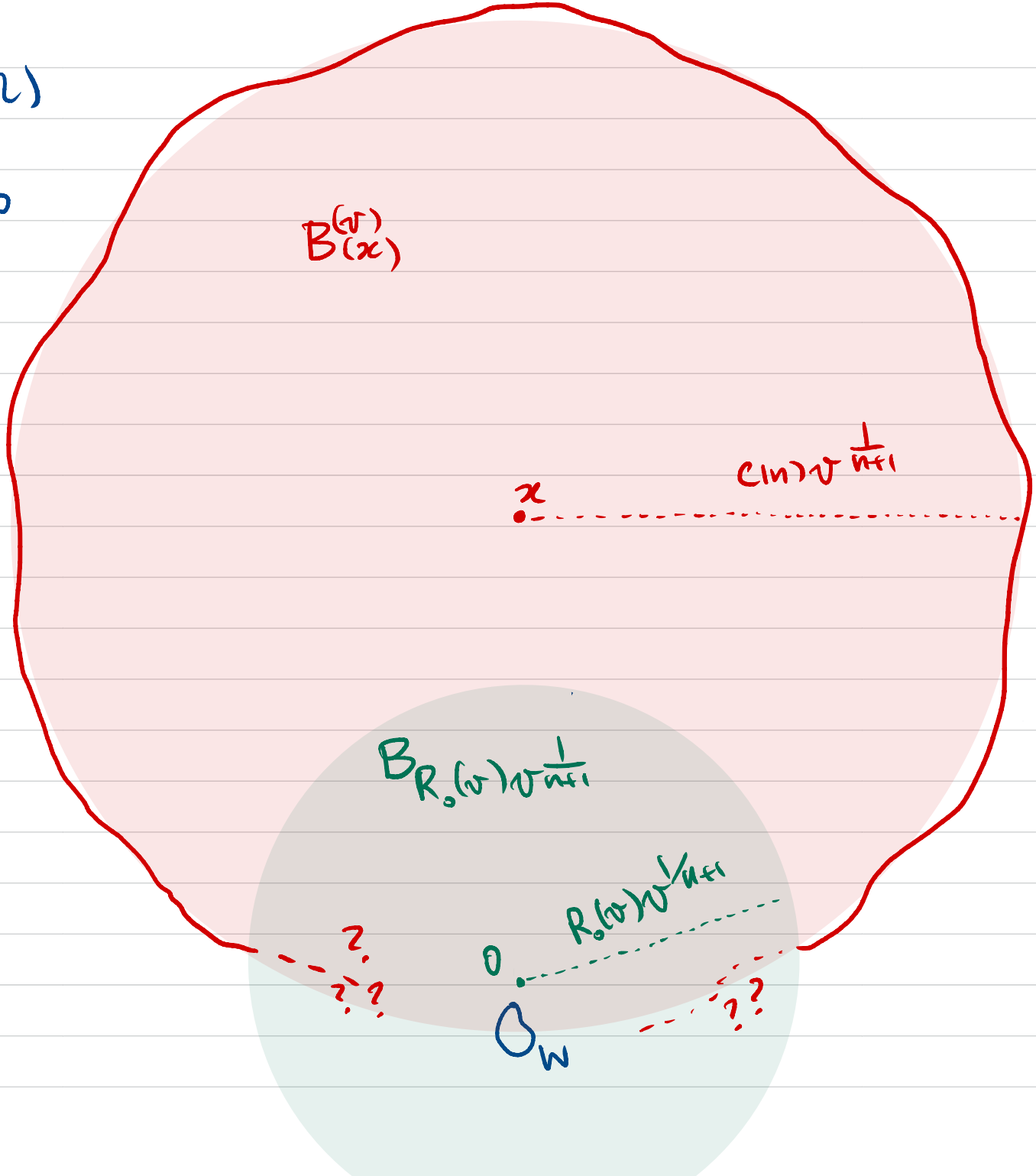
$E_v \approx$ BALL IF v LARGE

$H_{E_v} = O(v^{-1/n+1})$ IF $v \to +\infty$

BY QUANTITATIVE ISOP & IMPROVED CONVERGENCE

IF E_v S.T. $\psi_W(v) = P(E_v; \Omega)$

(1) $\dfrac{\psi_W(v)}{P(B^{(v)})} \longrightarrow 1$ AS $v \to \infty$



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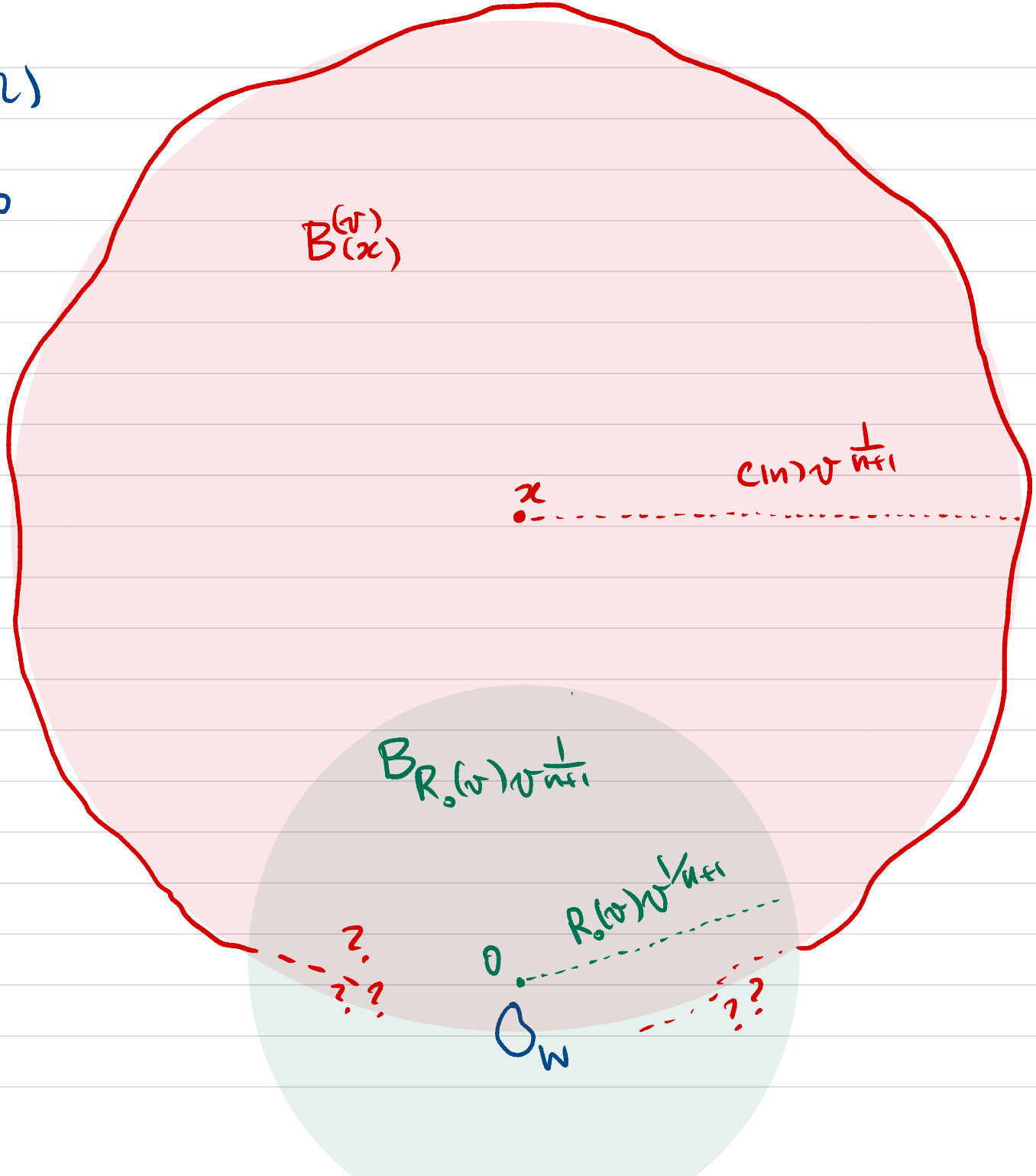
(2) $\dfrac{|E_v \Delta B^{(v)}(x)|}{v} \leq \dfrac{C(n)}{v^{1/2(n+1)}}$



IF E_v S.T. $\varphi_W(v) = P(E_v; \Omega)$

(1) $\dfrac{\varphi_W(v)}{P(B^{(v)})} \longrightarrow 1$ AS $v \to \infty$

(2) $\dfrac{|E_v \Delta B^{(v)}(x)|}{v} \leq \dfrac{C(n)}{v^{1/2(n+1)}}$

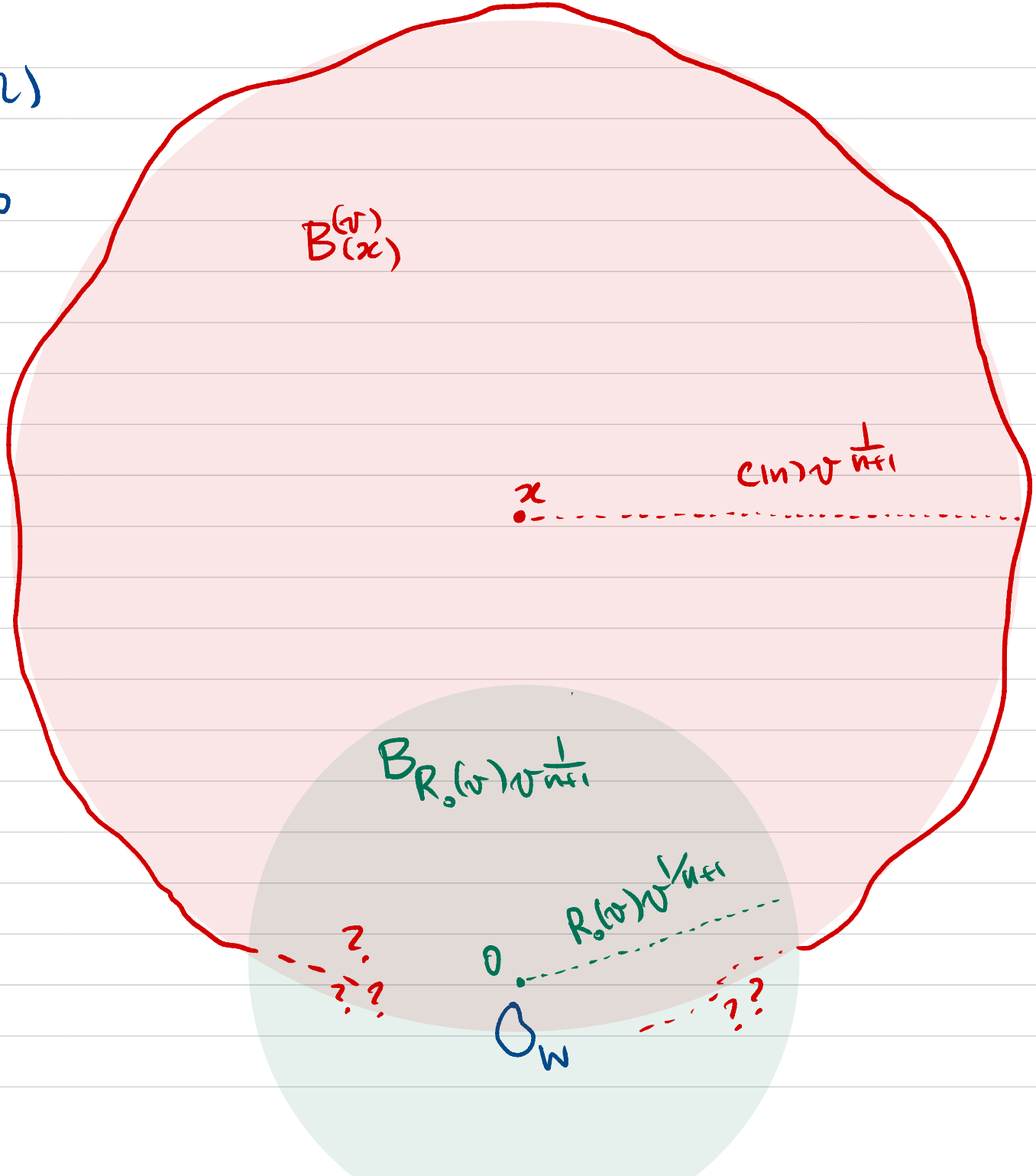
(3) $\exists\, R_0(v) \to 0^+ \quad (v \to +\infty)$

$R_0(v)\, v^{1/(n+1)} \longrightarrow +\infty$

S.T. $\partial E_v \setminus B_{R_0(v) v^{1/n+1}}$

IS C^1-SMALL NORMAL GRAPH

OVER $\partial B^{(v)}(x)$



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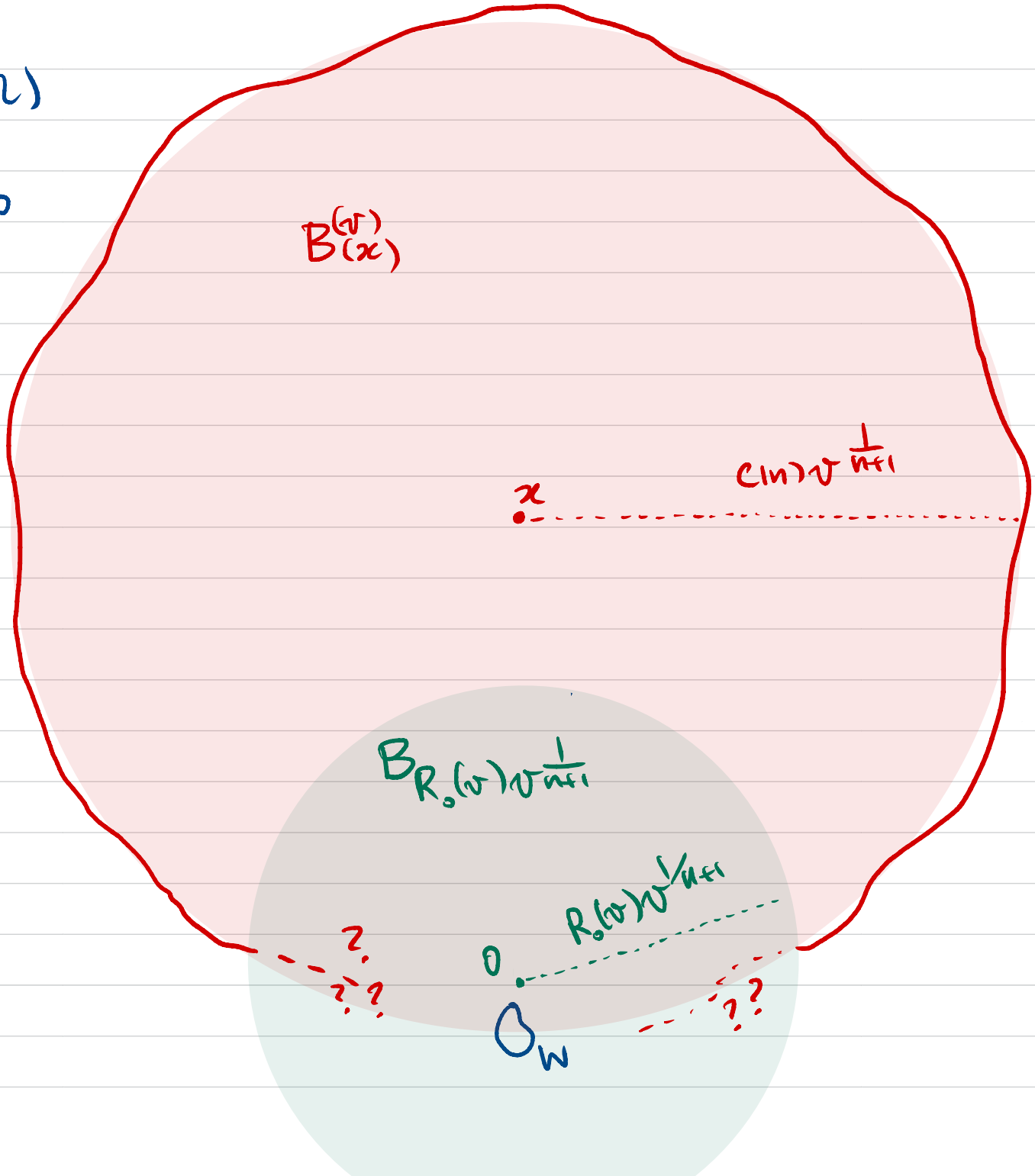
IS C^1-SMALL NORMAL GRAPH

OVER $\partial B^{(v)}(x)$

NO INFORMATION ON $\varphi_W(v)$

FEATURING W ITSELF!

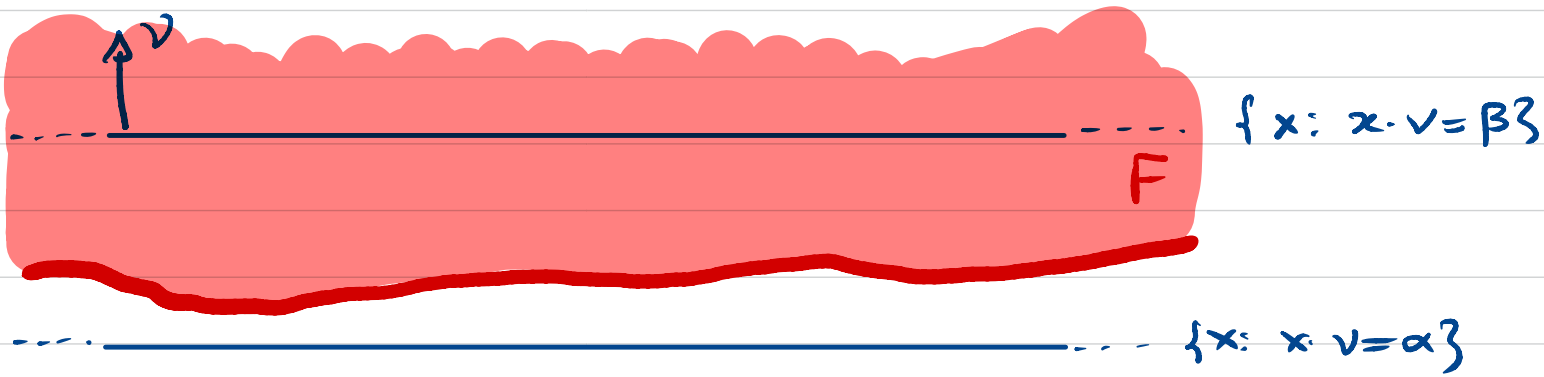


ISOPERIMETRIC RESIDUES

CONSIDER $\mathcal{F}$ = PAIRS (F,ν) S.T. $F \subseteq \mathbb{R}^{n+1}$, $\nu \in S^n$ &

1) $\mathrm{proj}_{\nu^\perp}(\partial F) = \nu^\perp$

2) $\partial F \subseteq \{x : \alpha < x \cdot \nu < \beta\}$ FOR SOME $\alpha, \beta \in \mathbb{R}$.



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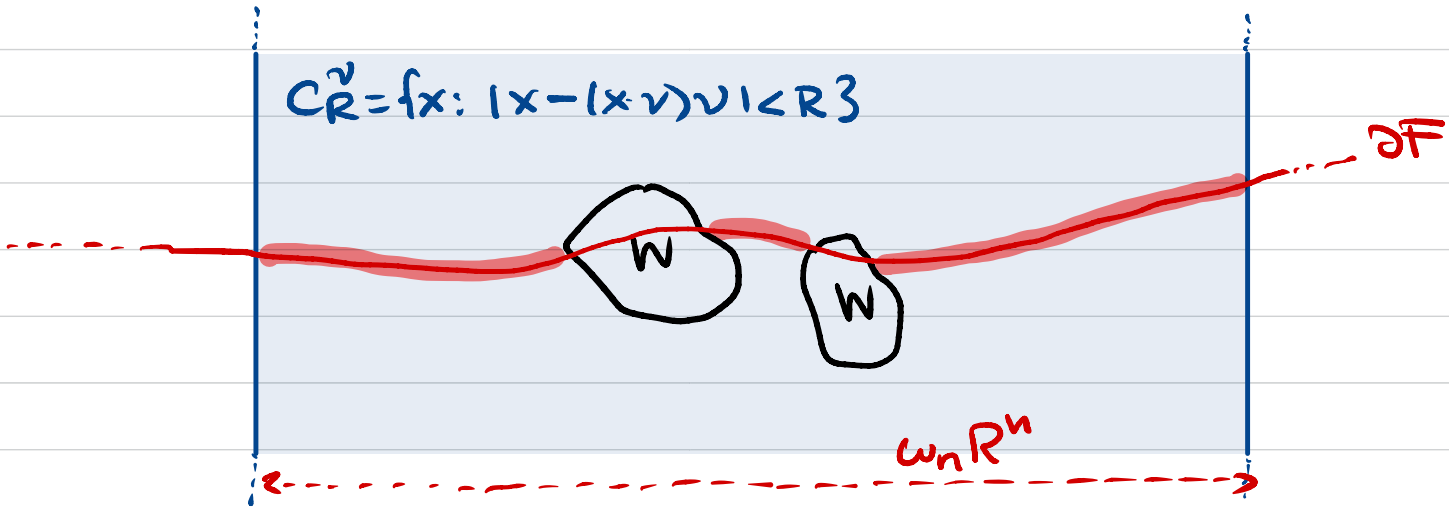
DEFINE W-RESIDUAL PERIMETER OF (F,ν)

$$\mathrm{res}_W(F,\nu) = \lim_{R\to\infty} \omega_n R^n - P(F; C_R^\nu \mid W)$$



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DEFINE ISOPERIMETRIC RESIDUE OF W

$$\mathcal{R}(W) = \sup\{\mathrm{res}_W(F,\nu) : (F,\nu) \in \mathcal{F}\}.$$

~ "PLATEAU'S PROB WITH FREE BOUNDARY ON W & AT ∞"

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~ "PLATEAU'S PROB WITH FREE BOUNDARY ON W & AT ∞"

~ CLOSE TO A SECTIONAL AREA OF W (BUT NOT QUITE THAT!)

ISOPERIMETRIC RESIDUES

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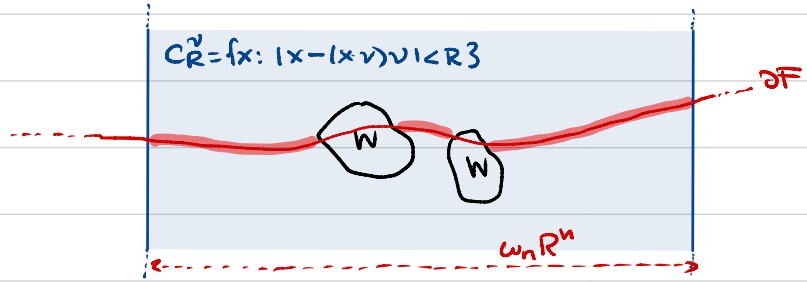

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THEN

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ISOPERIMETRIC RESIDUES

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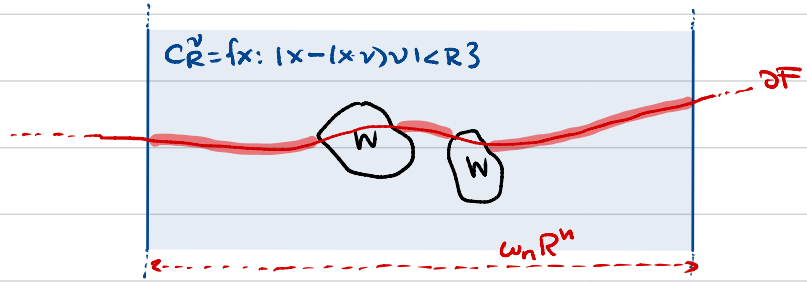

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2) $\exists\, R_2 = R_2(n,W)$ S.T. $\forall v_j \to +\infty$, $\forall E_j$ MINIMIZERS OF $\psi_W(v_j)$

THEN $\exists$ MAXIMIZER F OF $\mathcal{R}(W)$ S.T. (UP TO SUBS)

$$E_j \to F \quad \text{IN } L^1_{loc}(\mathbb{R}^{n+1})$$

& $(\forall R>0)$ $\partial E_j \cap (B_R \setminus B_{R_2}) \to \partial F \cap (B_R \setminus B_{R_2})$ IN C^1

ISOPERIMETRIC RESIDUES

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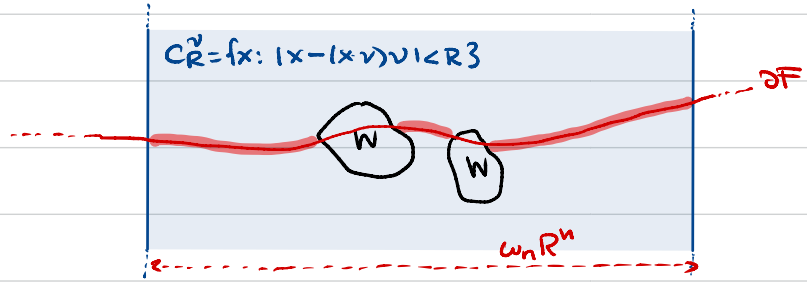

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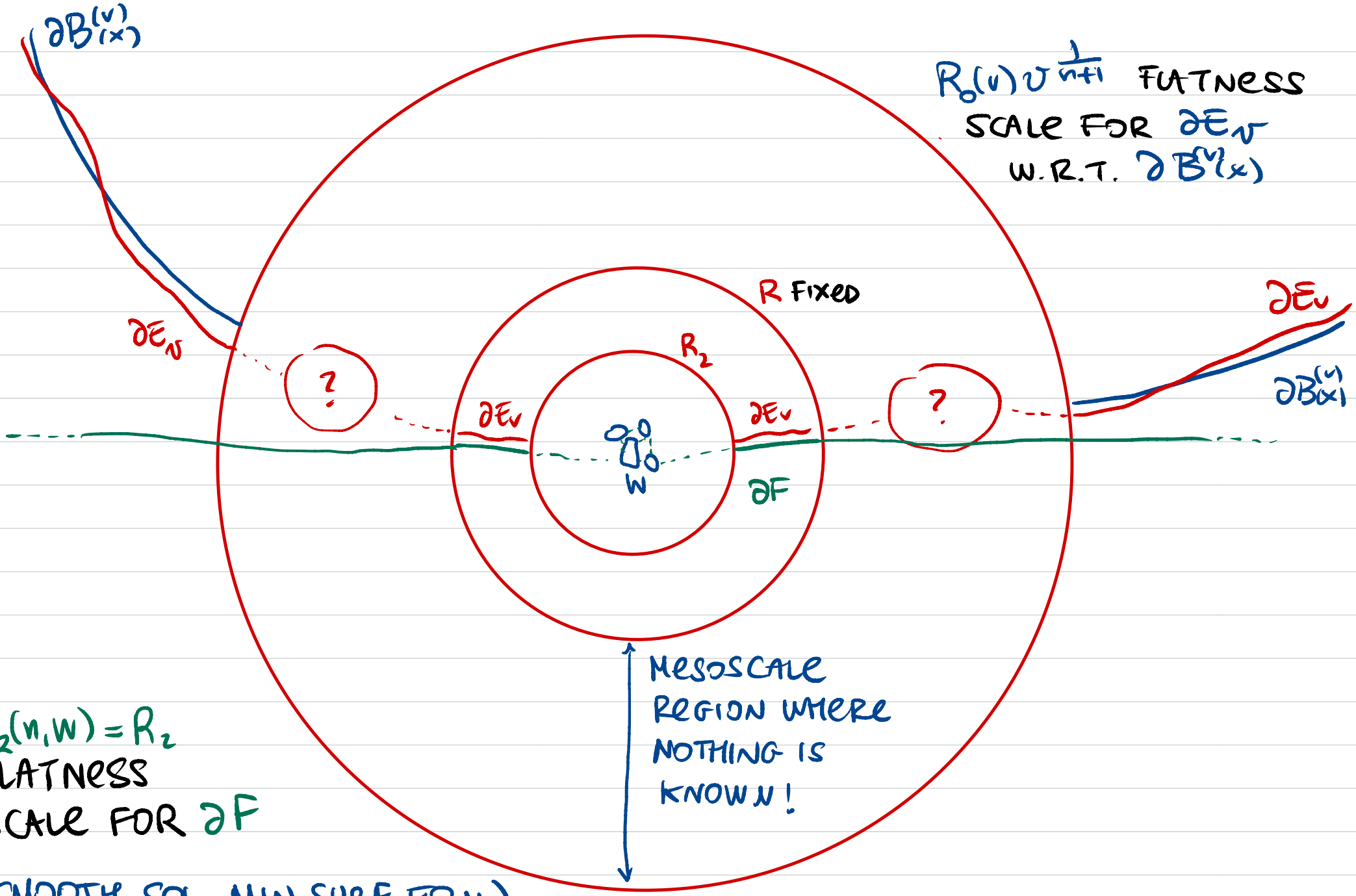
THEN $\exists$ MAXIMIZER F OF $\mathcal{R}(W)$ S.T. $(\partial E_j)\setminus B_{R_2} \to (\partial F)\setminus B_{R_2}$ IN C^1_{loc}

RMK: PROOFS OF 1) & 2) REQUIRE BLOW-DOWN & DECAY ANALYSIS

FOR EXTERIOR MINIMAL SURFACES (EXTERIOR FLATNESS CRITERION)

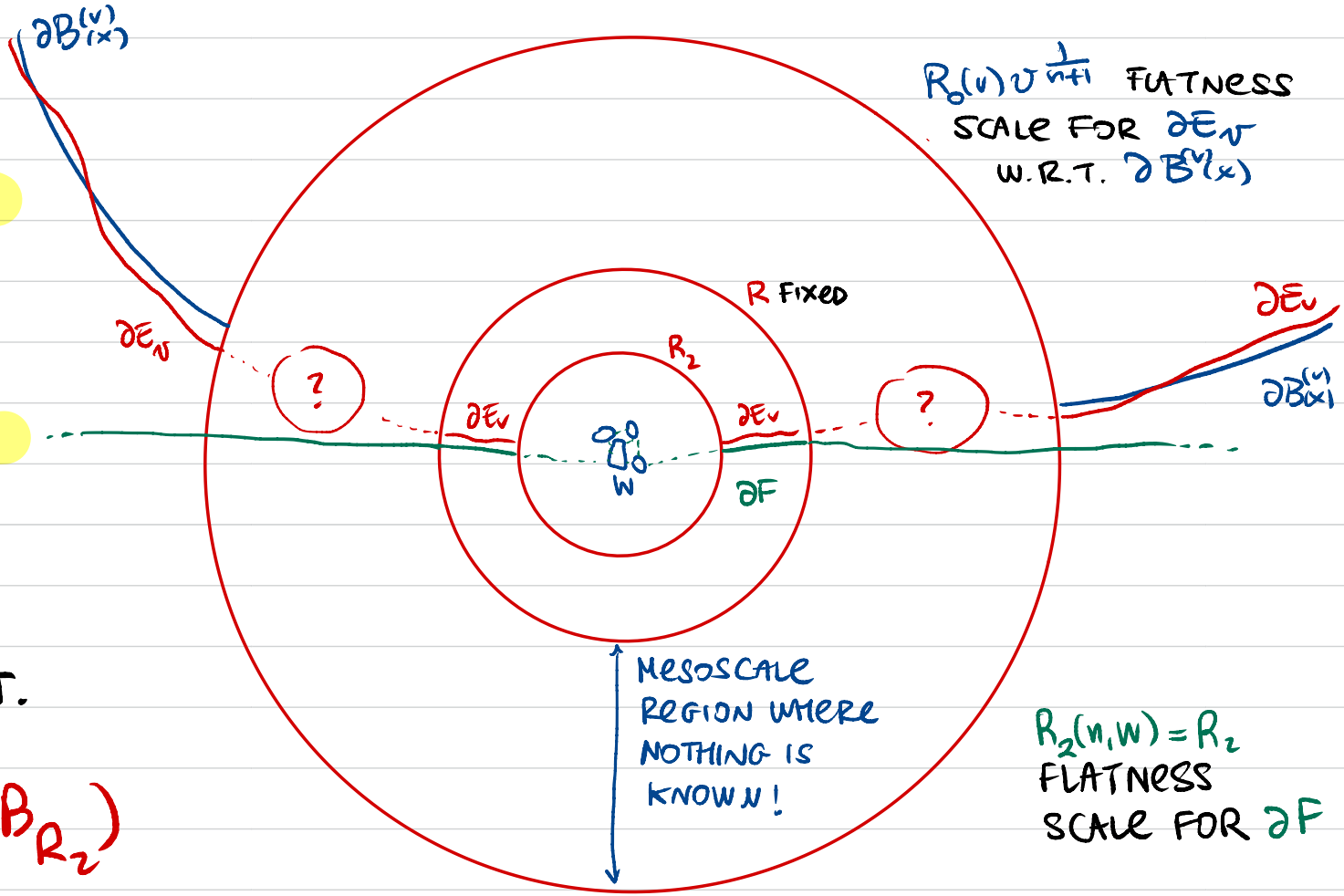


$R_2(n, W) = R_2$
FLATNESS SCALE FOR ∂F

(SMOOTH SOL. MIN. SURF. EQN.)

WE NEED A MESOSCALE FLATNESS CRITERION

TO DEDUCE MESOSCALE IMPROVED CONVERGENCE:



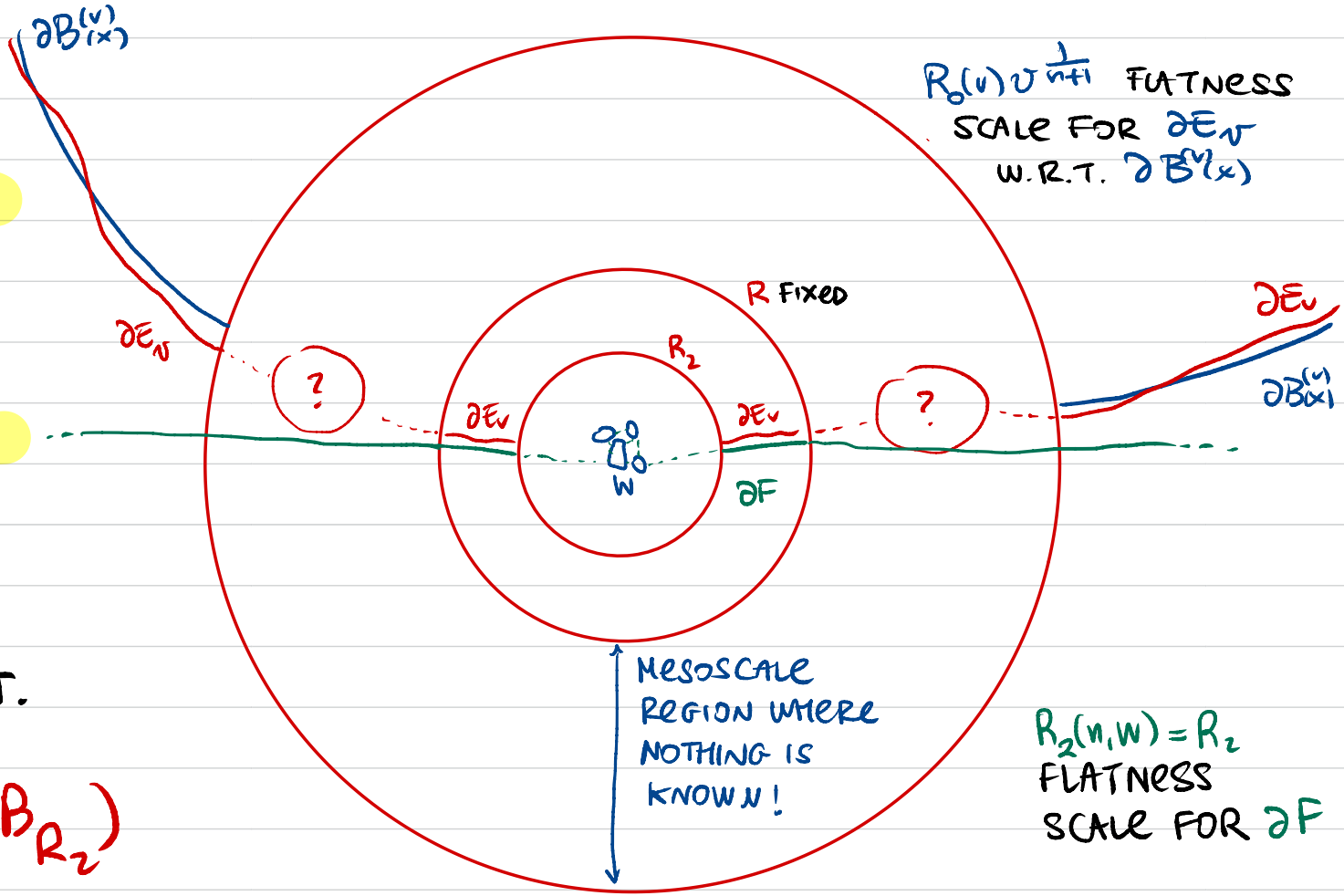
$\exists\, R_1 = R_1(n,W)$ S.T.

$\partial E_v \cap (B_{R_1 v^{\frac{1}{n+1}}} \setminus B_{R_2})$

$\subseteq C^1$-SMALL GRAPH OVER ∂F

WE NEED A MESOSCALE FLATNESS CRITERION

TO DEDUCE MESOSCALE IMPROVED CONVERGENCE:



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RMK: FOR v LARGE $R_1 v^{\frac{1}{n+1}} > R_0(v)\, v^{\frac{1}{n+1}} \Rightarrow$ OVERLAPPING!

MESOSCALE FLATNESS CRITERION $\Longrightarrow$

$\exists\, R_1 = R_1(n, W)$ S.T.

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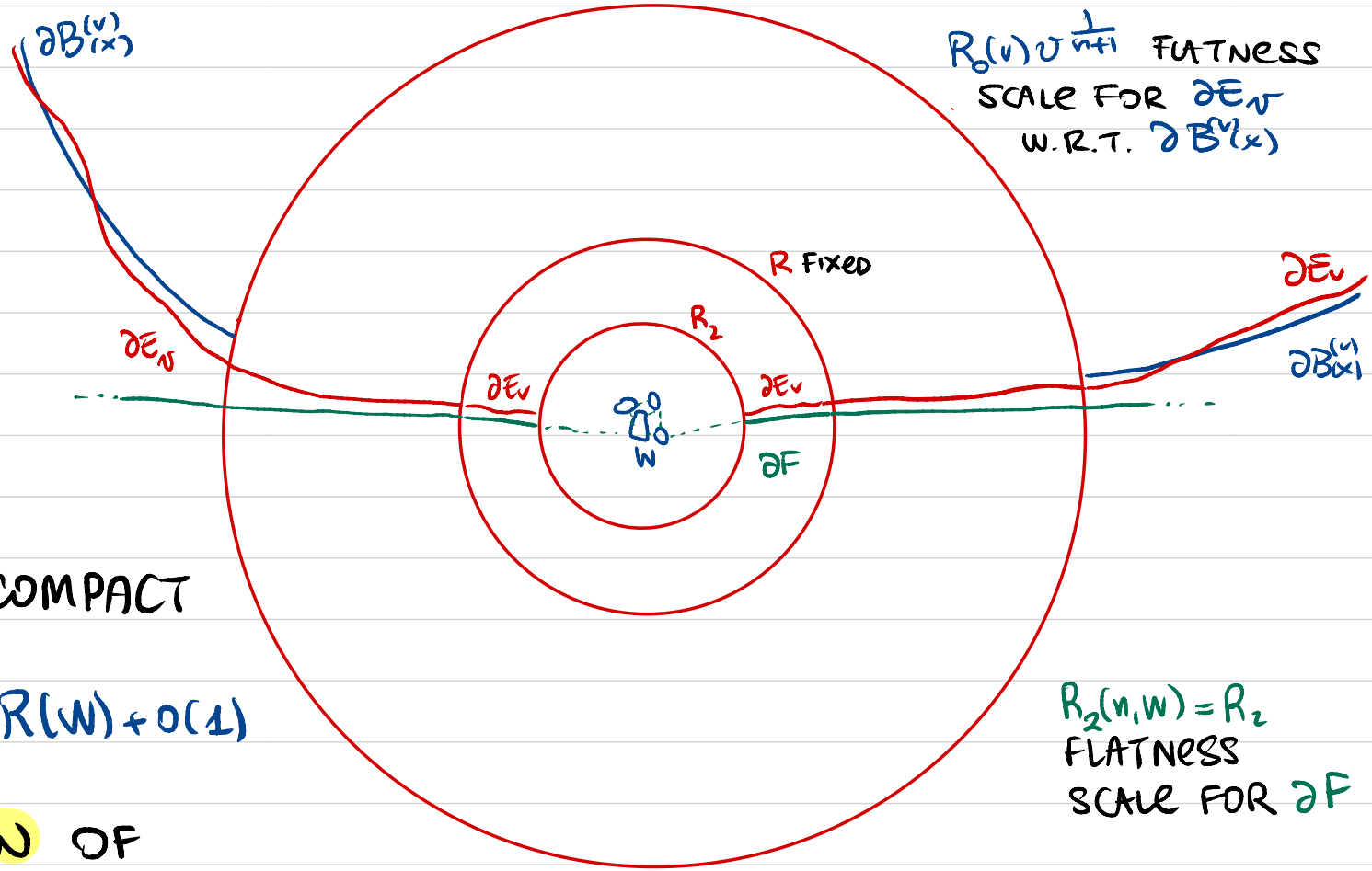
$\Downarrow$



THEOREM 1 IF W COMPACT

THEN $\psi_W(v) = P(B^{(v)}) - R(W) + o(1)$

& **FULL RESOLUTION** OF

EXTERIOR ISOPERIMETRIC SETS (UP TO W)

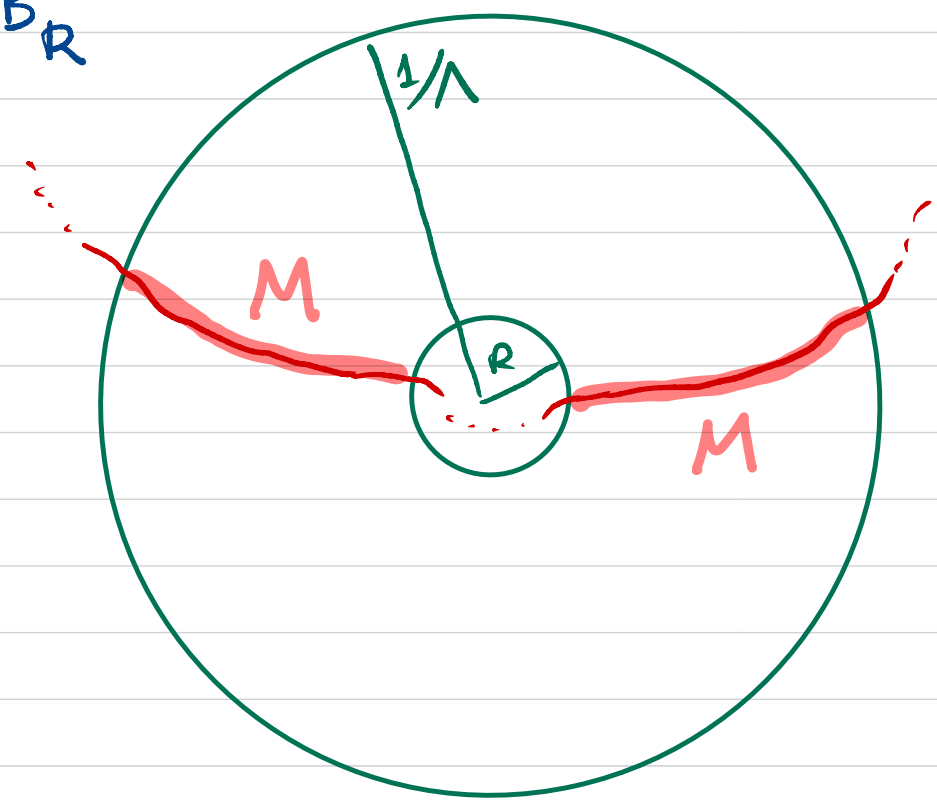
IN PARTICULAR, $\partial E_v \setminus B_{R_2}$ IS AN n-DIMENSIONAL DISK.

A MESOSCALE FLATNESS CRITERION

$\mathcal{M}_{\Lambda,R}$ = HYPERSURFACES M IN $\mathbb{R}^{n+1} \setminus B_R$

$|H_M| \leq \Lambda$ ON $B_{1/\Lambda} \setminus B_R$

$\mathrm{bdry}(M) \subseteq \partial B_R$



TWO NOTIONS OF FLATNESS

1) ANGULAR FLATNESS

2) AREA DEFICIT

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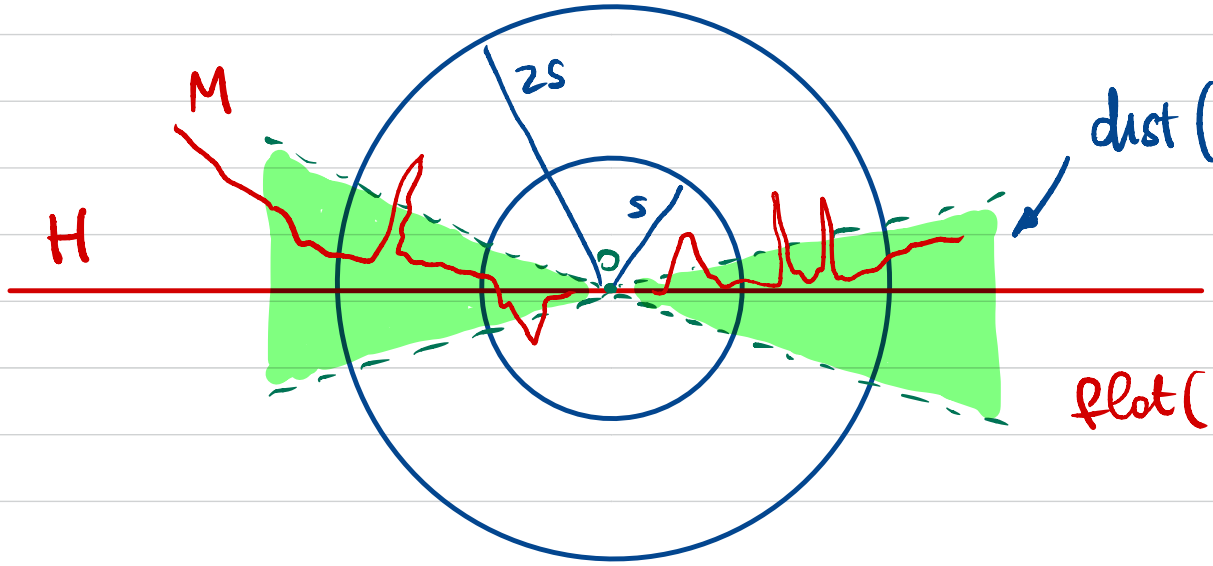
$M \in \mathcal{M}_{\Lambda,R}$ i.e. $|H_M| \le \Lambda$ ON $B_{1/\Lambda} \setminus B_R$ & $\mathrm{bdry}(M) \subseteq \partial B_R$

1) ANGULAR FLATNESS

$\mathcal{H}$ = HYPERPLANES THROUGH ORIGIN $\qquad H \in \mathcal{H}$

$$\mathrm{flat}(M,H,s) = \frac{1}{s^n} \int_{M \cap (B_{2s} \setminus B_s)} \mathrm{dist}^2_{S^n}\left(\frac{y}{|y|}, H \cap S^n\right) d\mathcal{H}^n(y).$$



$\mathrm{dist}\left(\frac{y}{|y|}, H \cap S^n\right) < \varepsilon$

$\mathrm{flat}(M,H,s) < \varepsilon$

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2) AREA DEFICIT: THE QUANTITY

$$\theta_{M,\Lambda,R}(r) = \frac{\mathcal{H}^n(M \cap (B_r \setminus B_R))}{r^n}$$

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SAME AS IN ALLARD IF $R = 0$!

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<u>RMK</u>: $\Theta_{M,\Lambda,R} \equiv \omega_n$ FOR HYPERPLANES $\quad \delta_{M,\Lambda,R}(r) = \omega_n - \Theta_{M,\Lambda,R}(r)$

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CRITERION $\forall \sigma \; \exists \; \varepsilon_0$: IF $\exists$ $s \gg R$, $s \ll 1/\Lambda$ (s = MESOSCALE)

SUCH THAT $\mathrm{flat}(M,H,s) \leq \varepsilon_0$ & $|\delta_{M,\Lambda,R}(s)| \leq \varepsilon_0$

THEN $M \cap (B_{s_*} \setminus B_s)$ is σ-FLAT IN C^1 W.R.T. H

$$s_* = \max\left\{\frac{\varepsilon_0}{\Lambda}, R_*\right\} \qquad R_* = \sup\left\{\rho \geq s : \delta_{M,\Lambda,R}(\rho) \geq -\varepsilon_0\right\}.$$

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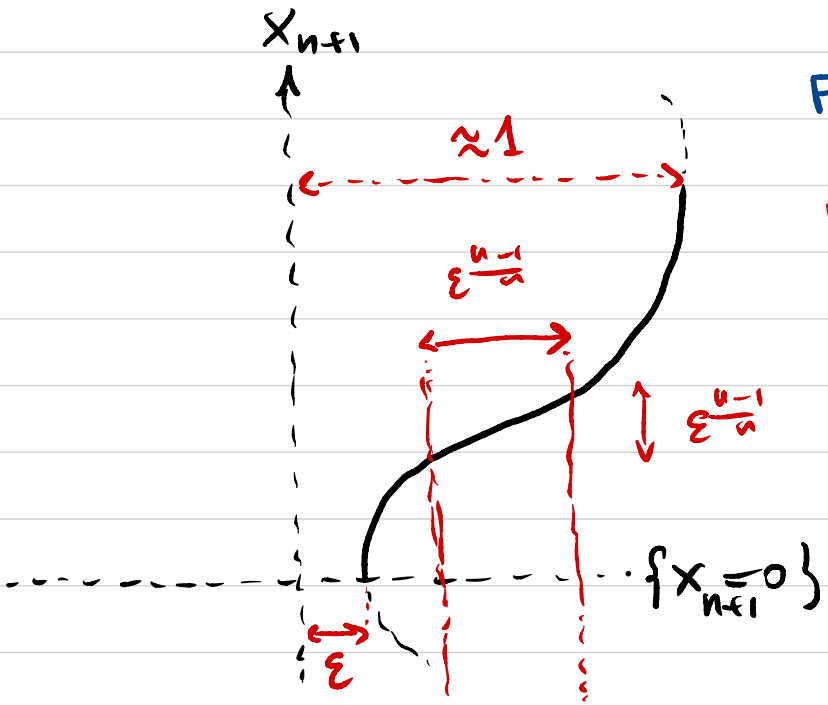
EXAMPLE UNDULOIDS CMC $= n$ (LIKE UNITBALLS) & WAIST SIZE ε



FLATNESS W.R.T. $\{x_{n+1}=0\}$

$|\nabla f|(r) \approx O(\varepsilon^{(n-1)/n})$ FOR $r = O(\varepsilon^{(n-1)/n})$

$$1 \gg \varepsilon^{\frac{n-1}{n}} \gg \varepsilon \qquad (n \geq 2)$$

THEOREM 2 $W \subseteq \mathbb{R}^{n+1}$ COMPACT

1) $$\mathcal{S}(W) \leq \mathcal{R}(W) \leq \mathcal{P}(W)$$

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RMK: $\mathcal{R}(\lambda W) = \lambda^n \mathcal{R}(W) \quad \forall \lambda > 0$

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THEOREM 2 IF $W \subseteq \mathbb{R}^{n+1}$ COMPACT, THEN

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4) $\exists\, R_2 = R_2(n, W)$ S.T. IF (F, ν) MAXIMIZER OF $\mathcal{R}(W)$, THEN

$\partial F \setminus B_{R_2}$ = GRAPH OVER $\nu^\perp$ OF f SOL MIN SURF EQN

$f(x) \equiv a \qquad (n=1)$

$$\left| f(x) - \left(a + \frac{b}{|x|^{n-2}} + \frac{c \cdot x}{|x|^n} \right) \right| \leq \frac{C_0}{|x|^n} \qquad (n \geq 2)$$

$$|x|^{n-1} |\nabla f(x)|, \; |x|^n |\nabla^2 f(x)| \leq C_0$$