THREE LECTURES ON ISOPERIMETRIC & PLATEAU-TYPE PROBLEMS

FRANCESCO MAGGI

UNIVERSITY OF TEXAS AT AUSTIN

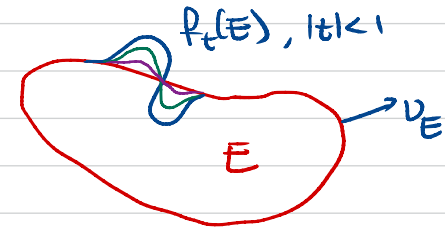
GEOMETRIC ANALYSIS & PDE on GARDA LAKE

8-10 JUNE 2022

LECTURE 2

BUBBLING WITH ALMOST CONSTANT MEAN CURVATURE & ALMOST MINIMAL SURFACES

VOLUME PRESERVING VARIATIONS



GIVEN $E \subseteq \mathbb{R}^n$, $\forall X \in C_c^\infty(\mathbb{R}^n; \mathbb{R}^n)$ WITH $\underbrace{\int_{\partial E} X \cdot \nu_E = 0}$,

"ZERO-AVERAGE BOUNDARY MOTION"

ONE CAN FIND DIFFEOMORPHISMS $f_t : \mathbb{R}^n \to \mathbb{R}^n$

SUCH THAT $|f_t(E)| = |E| \quad \forall |t| < t_0$ (VOLUME PRESERVED)

& $f_t(x) = x + t\,X(x) + O(t^2)$ IN $C^1(\mathbb{R}^n)$ AS $t \to 0^+$

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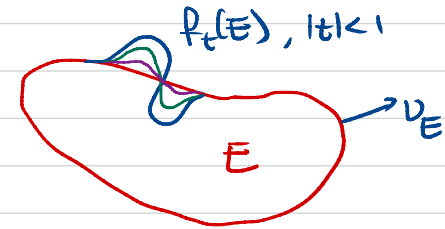

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& $\quad f_t(x) = x + t\,X(x) + O(t^2)$ IN $C^1(\mathbb{R}^n)$ AS $t \to 0^+$

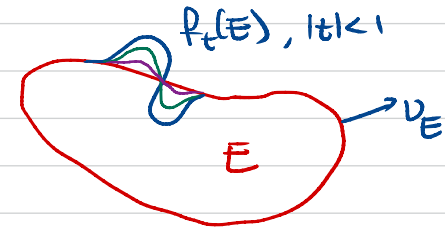
PERIMETER EXPANSION

SINCE $P(E) = \mathcal{H}^{n-1}(\partial E)$, BY THE AREA FORMULA,

$$\underbrace{P(f_t(E))}_{\mathcal{H}^{n-1}(f_t(\partial E))} = \int_{\partial E} J^{\partial E} f_t \, d\mathcal{H}^{n-1} = \int_{\partial E} \left(1 + t\, \mathrm{div}^{\partial E} X + O(t^2)\right) d\mathcal{H}^{n-1} = P(E) + t \int_{\partial E} \mathrm{div}^{\partial E} X + O(t^2)$$

$$\partial [f_t(E)] = f_t(\partial E)$$

VOLUME PRESERVING VARIATIONS



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FIRST VARIATION FORMULAE

TANGENTIAL DIV. THEOREM

$$\frac{d}{dt}\Big|_{t=0} P(f_t(E)) = \int_{\partial E} dw^{\partial E}\, X = \int_{\partial E} (X \cdot \nu_E)\, H_E$$

& $\frac{d}{dt}\Big|_{t=0} \int_{f_t(E)} g = \int_{\partial E} (X \cdot \nu_E)\, g$ BY SIMILAR COMPUTATIONS

ALMOST CMC BOUNDARIES: MOTIVATION

CAPILLARITY ENERGY $\mathcal{F}_g(E) = P(E) + \int_E g(x)\,dx$

ABSOLUTE & LOCAL MINIMIZERS OF $\mathcal{F}_g$ (AT FIXED VOLUME) SATISFY

THE CRITICAL POINT CONDITION

$$\frac{d}{dt}\Big|_{t=0} \mathcal{F}_g(f_t(E)) = 0 \qquad \forall X \in C_c^\infty(\mathbb{R}^n;\mathbb{R}^n) \text{ s.t. } \int_{\partial E} X \cdot \nu_E = 0$$

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$$\iff \int_{\partial E} (X\cdot\nu_E)(H_E + g) = 0 \qquad \forall X \in C_c^\infty(\mathbb{R}^n;\mathbb{R}^n) \text{ s.t. } \int_{\partial E} X\cdot\nu_E = 0$$

φ

$\varphi \in C_c^\infty(\mathbb{R}^n)$

$X = \varphi\,\nu_E$ on $\partial E \to \mathbb{R}^n$

EXTEND TO $C_c^\infty(\mathbb{R}^n;\mathbb{R}^n)$

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$$\Longleftrightarrow \quad \int_{\partial E} (X\cdot\nu_E)(H_E + g) = 0 \qquad \forall X \in C_c^\infty(\mathbb{R}^n;\mathbb{R}^n) \text{ s.t. } \int_{\partial E} X\cdot\nu_E = 0$$

$\Longleftrightarrow$ $\exists \lambda \in \mathbb{R}$ s.t. $H_E + g = \lambda$ ON ∂E — EULER-LAGRANGE EQUATION

λ CAN BE COMPUTED AS FOLLOWS

ALMOST CMC BOUNDARIES: MOTIVATION

COMPUTING λ S.T. $H_E + g = \lambda$ ON ∂E

RECALL THAT $\operatorname{div}(x) = n$ & $\operatorname{div}^T x = k$ IF T k-DIMENSIONAL PLANE

$$\operatorname{div}^T Z = \sum_{i=1}^{k} \tau_i \cdot \nabla_{\tau_i} Z$$

IF $\{\tau_i\}_{i=1}^{k}$ ORTHONORMAL BASIS OF T

ALMOST CMC BOUNDARIES: MOTIVATION

COMPUTING λ S.T. $H_E + g = \lambda$ ON ∂E

RECALL THAT $\operatorname{div}(x) = n$ & $dw^T x = k$ IF T k-DIMENSIONAL PLANE

WE FIND

$$(n-1)\,P(E) = \int_{\partial E} dw^T x$$

ALMOST CMC BOUNDARIES: MOTIVATION

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WE FIND

$$(n-1)P(E) = \int_{\partial E} dw^T x \overset{\text{TANG. DIV. THM}}{=} \int_{\partial E} x \cdot (H_E \nu_E)$$

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$$\underset{\text{DIV. THM.}}{=} \int_E \operatorname{div}((\lambda - g)x)$$

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$$\underset{\text{DIV. THM.}}{=} \int_E \operatorname{div}((\lambda - g)x) = n|E|\lambda - \int_E \operatorname{div}(x\, g(x))$$

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$$\underset{\text{DIV. THM.}}{=} \int_E d_w((\lambda - g)x) = n|E|\lambda - \int_E \operatorname{div}(x\, g(x))$$

$$\Rightarrow \boxed{\lambda = \underbrace{\frac{(n-1)P(E)}{n|E|}}_{=: H_E^0} + \frac{1}{n} \fint_E d_w(x\, g(x))}$$

ALMOST CMC BOUNDARIES: MOTIVATION

IF E CRITICAL POINT OF $\mathcal{F}_g$ WITH FIXED VOLUME

THEN

$$H_E + g = H_E^0 + \frac{1}{n} \int_E dw\,(x g(x)) \quad \text{ON } \partial E$$

$$H_E^0 = \frac{(n-1)P(E)}{n|E|}$$

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NOW: $$\left| \frac{1}{n} \fint_E dw\,(xg(x)) \right| \leq C(n,g)$$

HENCE $$\left\| \frac{H_E}{H_E^0} - 1 \right\|_{C^0(\partial E)} \leq \frac{C(n,g)}{H_E^0} \leq C(n,g) \frac{v}{P(E)} \leq C(n,g)\, v^{\frac{1}{n}}$$

BY ISOPERIMETRY ($P(E) \geq c(n)\, v^{\frac{n-1}{n}}$)

ALMOST CMC BOUNDARIES: MOTIVATION

IF E CRITICAL POINT OF $\mathcal{F}_g$ WITH FIXED VOLUME

THEN $\delta_{CMC}(E) \leq C(n,g)\, v^{1/n}$

WHERE BY DEFINITION $\delta_{CMC}(E) = \left\| \frac{H_E}{H_E^0} - 1 \right\|_{C^0(\partial E)}$

$$H_E^0 = \frac{(n-1)P(E)}{n|E|}$$

SO WE LOOK AT BOUNDARIES WITH ALMOST CMC

ALMOST CMC BOUNDARIES: MOTIVATION

IF E CRITICAL POINT OF $\mathcal{F}_g$ WITH FIXED VOLUME

THEN $H_E + g = H_E^0 + \frac{1}{n} \fint_E dw\,(xg(x))$ ON ∂E, $H_E^0 = \frac{(n-1)P(E)}{n|E|}$

HENCE

$$\delta_{CMC}(E) = \left\| \frac{H_E}{H_E^0} - 1 \right\|_{C^0(\partial E)} \leq C(n,g)\, v^{1/n}$$

I.E. CRITICAL POINTS OF $\mathcal{F}_g$ WITH SMALL VOLUME HAVE δ_{CMC} SMALL

WE THUS STUDY BOUNDARIES WITH ALMOST CONSTANT MEAN CURVATURE

ALMOST CMC BOUNDARIES & BUBBLING

CONTROL PARAMETER	"$\delta = 0$"
$\delta_{ISO}(E) = \dfrac{P(E)}{n\omega_n^{1/n}\lvert E\rvert^{\frac{n-1}{n}}} - 1$	$E = B_r(x)$ (ISOP. THEOREM)
$\delta_{CMC}(E) = \left\lVert \dfrac{H_{\partial E}}{H^0_{\partial E}} - 1 \right\rVert_{C^0(\partial E)}$	$E = B_r(x)$ (ALEXANDROV THM)

ALMOST CMC BOUNDARIES & BUBBLING

CONTROL PARAMETER	"$\delta = 0$"	SHARP QUANTITATIVE FORM
$\delta_{iso}(E) = \dfrac{P(E)}{n\omega_n^{1/n}\lvert E\rvert^{\frac{n-1}{n}}} - 1$	$E = B_r(x)$ (ISOP. THEOREM)	$C(n)\sqrt{\delta_{iso}(E)} \geq \inf_{x\in\mathbb{R}^n} \left(\dfrac{\lvert E\Delta B_r(x)\rvert}{\lvert E\rvert}\right)^2$ (FMP06)
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THE KEY DIFFICULTY IS BUBBLING!!!

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THE KEY DIFFICULTY IS BUBBLING!!!

"QUALITATIVE" STABILITY $\forall \varepsilon > 0 \ \exists \delta > 0$ S.T.

$\delta_{iso}(E) < \delta \Rightarrow \exists x \in \mathbb{R}^n$ S.T. $\lvert E \Delta B_r(x)\rvert \leq \lvert E\rvert \varepsilon$

$r = \left(\lvert E\rvert / \omega_n\right)^{1/n}$

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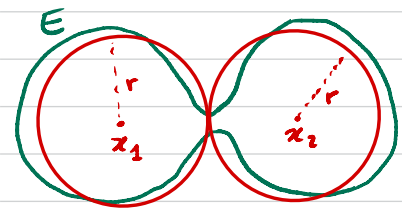

$k=2$

"QUALITATIVE" STABILITY $\forall \varepsilon > 0 \ \exists \delta > 0$ S.T.

$\delta_{CMC}(E) < \delta \Rightarrow \exists k \geq 1$, $\exists x_1 \dots x_k \in \mathbb{R}^n$ S.T. $\left\lvert E \Delta \bigcup_{j=1}^{k} B_r(x_j)\right\rvert \leq \lvert E\rvert \varepsilon$ $\quad r = \left(\lvert E\rvert / k\omega_n\right)^{1/n}$

WITH $\lvert x_j - x_{j+1}\rvert = 2r$ IF E CONNECTED

BUBBLING IF $k \geq 2$

INTERMISSION

WE HAVE JUST INTRODUCED BUBBLING BY LOOKING AT $\delta_{CMC}(E_j) \to 0$

WHERE δ_{CMC} = C^0- OSCILLATION OF MEAN CURVATURE

NOTHING WORST CAN HAPPEN EVEN WITH WILDER OSCILLATIONS!

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THM (DELGADINO-M.'17)

IF $\{E_j\}_j$, E ARE SUCH THAT $|E_j \Delta E| \to 0$, $P(E_j) \to P(E)$, $0 < |E| < \infty$,

& $\exists\, \lambda \in \mathbb{R}$ S.T. $\displaystyle\int_{\partial E_j} \mathrm{div}^{\partial E_j} X - \lambda \int_{\partial E_j} X \cdot \nu_{E_j} \to 0 \quad \forall\, X \in C^1_c(\mathbb{R}^n; \mathbb{R}^n)$

THEN $\lambda = \dfrac{(n-1)P(E)}{n|E|}$ & E = FINITE UNION OF BALLS OF RADIUS $\dfrac{n}{\lambda}$

TWO STRAINS OF PAPERS ABOUT QUANTITATIVE ANALYSIS OF ALMOST CMC

1) PAPERS WITH A BUBBLING-PREVENTING ASSUMPTION

2) PAPERS WHICH ALLOW FOR (& QUANTIFY) BUBBLING

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FOR $E \subseteq \mathbb{R}^n$ CONNECTED WITH $H_E^0 = \frac{(n-1)P(E)}{n|E|} = n-1$ (WLOG BY SCALING)

CIRAOLO-VEZZONI '15 UNDER EXTERIOR BALL CONDITION OF RADIUS ρ

$\Rightarrow \partial E = \{x + u(x)\nu_{B_1(x)} : x \in \partial B_1\}$ & $\delta_{CMC}(E) \geq c(n,\rho)\|u\|_{C^1(\partial B_1)}$

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KRUMMEL-M. '16 UNDER PERIMETER BOUND $P(E) \leq 2\rho P(B_1)$ ($0<\rho<1$ FIXED)

$\Rightarrow$ SAME CONCLUSION AS [CV15]

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MAGNANINI-POGGESI '16,'17,'18,'21 UNDER INTERIOR & EXTERIOR BALL CONDITION (*)

SEVERAL ESTIMATES ON $r_E^{out} - r_E^{in}$ IN TERMS OF $\left\| \frac{H_{\partial E}}{H^0_{\partial E}} - 1 \right\|_{L^p(\partial E)} = \delta^{(p)}_{CMC}(E)$

(SOME OF THEM SHARP) $p \in [1,\infty)$.

(*) BLOCKS GRAPHICALITY/EMBEDDEDNESS SCALE!

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2) PAPERS WHICH ALLOW FOR & QUANTIFY BUBBLING

BREZIS-CORON '84 IN $\mathbb{R}^3$ WITH $\partial E = u(B^2)$, $u: B^2 \to \mathbb{R}^3$ CONFORMAL

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CIRAOLO-M. '15 IN $\mathbb{R}^n$, E BOUNDED CONNECTED, $H_E^0 = \frac{(n-1)P(E)}{n|E|} = n-1$, $H_E > 0$,

THEN $\exists\, k \geq 1$ $\exists\, x_1 \dots x_k \in \mathbb{R}^n$ S.T.

$$|E \Delta \bigcup_{j=1}^{k} B_1(x_j)| \leq C(n) k^2 |E| \cdot \delta_{CMC}(E)^{1/2(n+1)}$$

$$|P(E) - kP(B_1)| \leq \text{SAME AS PREVIOUS ONE}$$

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$$|P(E) - kP(B_1)| \leq \text{SAME AS PREVIOUS ONE}$$

$$\left(G = \bigcup_{j=1}^{k} B_1(x_j)\right) \qquad hd(\partial E, \partial G) \leq C(n)\, k^{3/(n-1)}\, \mathrm{diam}(E)\, \delta_{CMC}(E)^{1/8(n-1)^2 n(n+1)}$$

∂E IS QUANT. C^1-CLOSE TO ∂G UP TO DISTANCE

$C(n)\, \delta(E)^{1/8n(n+1)}$ FROM TANGENCY POINTS.

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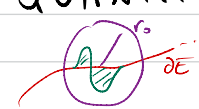
$$|E \Delta \bigcup_{j=1}^{k} B_1(x_j)| \leq C(n) k^2 |E| \cdot \delta_{CMC}(E)^{1/2(n+1)} \qquad [\dots]$$

COROLLARY IF E LOCAL MINIMIZER OF $\mathcal{F}_g$ WITH $|E|$ SMALL

THEN BUBBLING CANNOT OCCUR (& SHARPER ESTIMATES HOLD!)

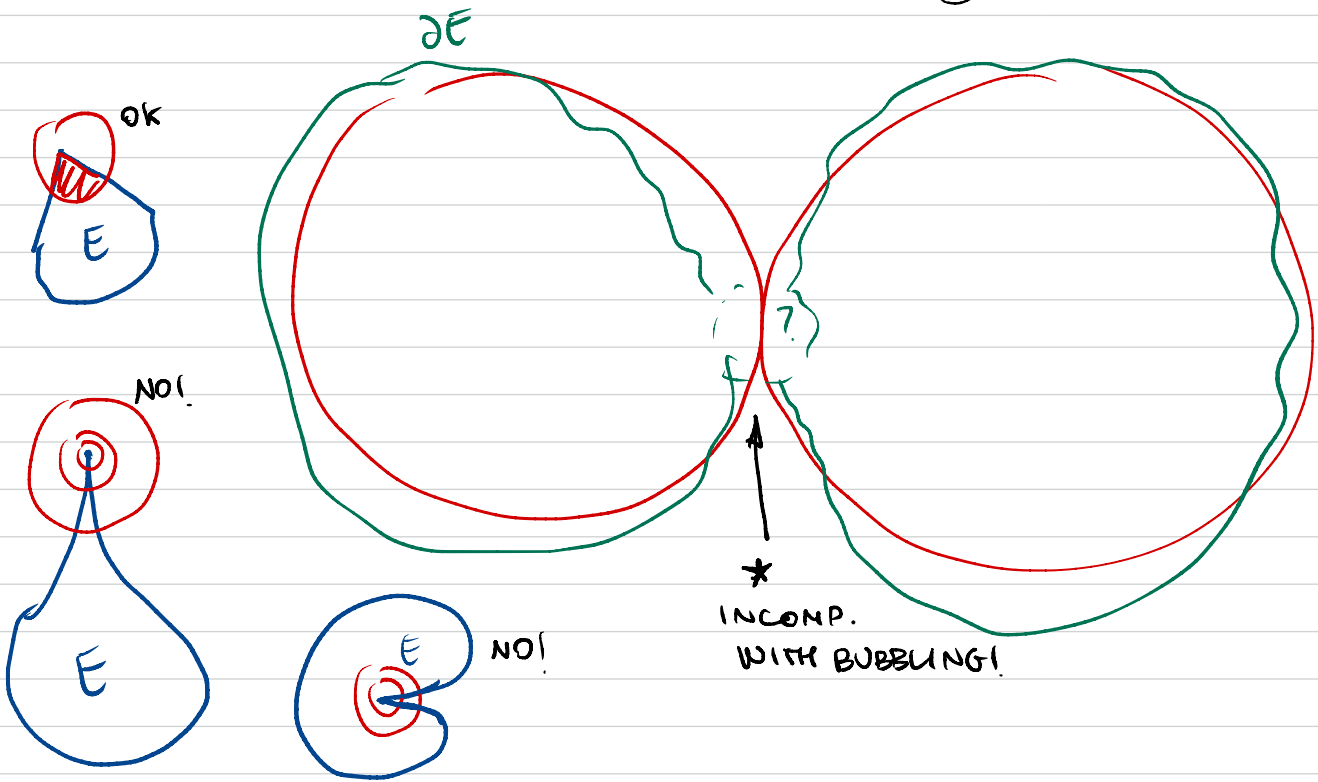
PROOF: VOLUME DENSITY ESTIMATES ARE NOT COMPATIBLE WITH TANGENCY POINTS

RMK: LOCAL MIN. CANNOT BE COMPARED TO BALLS $\Rightarrow$ CANNOT USE QUANTITATIVE ISOPERIMETRY!



$c(n) = \lambda \omega_n \quad \lambda \in (0,1)$

VOL DENSITY ESTIMATES: $c(n) r^n < |E \cap B_r(x)| < (\omega_n - c(n)) r^n, \quad \forall\, x \in \partial E$ $\circledast$



TWO STRAINS OF PAPERS ABOUT QUANTITATIVE ANALYSIS OF ALMOST CMC

2) PAPERS WHICH ALLOW FOR & QUANTIFY BUBBLING

BREZIS-CORON '84 IN $\mathbb{R}^3$ WITH $\partial E = u(B^2)$, $u: B^2 \to \mathbb{R}^3$ CONFORMAL

CIRAOLO-M. '15 IN $\mathbb{R}^n$, E BOUNDED CONNECTED, $H_E^0 = \frac{(n-1)P(E)}{n|E|} = n-1$, $H_E > 0$,

THEN $\exists\, k \geq 1$ $\exists\, x_1 \dots x_k \in \mathbb{R}^n$ S.T.

$$|E \Delta \bigcup_{j=1}^{k} B_1(x_j)| \leq C(n) k^2 |E| \cdot \delta_{CMC}(E)^{1/2(n+1)} \qquad [\dots]$$

JULIN - NIINIKOSKI '20 QUANTITATIVE ESTIMATES WITH $\int_{\partial E} |H_E - \lambda|^{n-1} d\mathcal{H}^{n-1}$

IN PLACE OF $\delta_{CMC}(E)$ (NO $H_E > 0$ ON ∂E) & APPLICATIONS TO THE

VOLUME PRESERVING MEAN CURVATURE FLOW IN $\mathbb{R}^3$

ALMOST-MINIMAL SURFACES : MOTIVATION

MINIMAL SURFACES ARISE AS MODELS FOR SOAP FILMS

AT MECHANICAL EQUILIBRIUM HANGING FROM A WIRE FRAME $W \subseteq \mathbb{R}^3$

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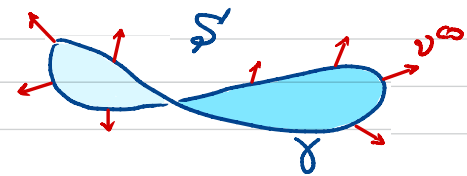
LAPLACE/YOUNG 1806 $M \subseteq \mathbb{R}^3$ SURFACE



THE TOTAL FORCE ON A SURFACE ELEMENT $S \subseteq M$

IS DUE TO A UNIFORM TANGENTIAL PULL ALONG $\gamma = bd(dS)$

S.T. FORCE $= \sigma \int_\gamma \nu_S^\infty = \sigma \int_S H_M \nu_M$ WHERE H_M MEAN CURVATURE W.R.T. ν_M

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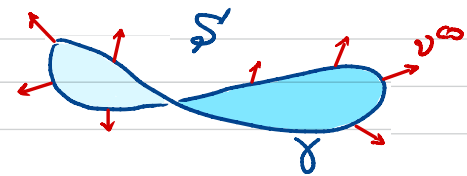

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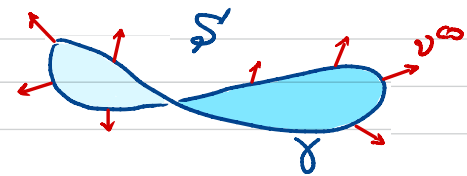

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EQUILIBRIUM $H_M = 0$

ALMOST-MINIMAL SURFACES : MOTIVATION

$$\begin{cases} H_M = 0 \\ \partial M = W \end{cases}$$

HAS NO LENGHT SCALE : λM WILL SOLVE FOR λW

NO MATTER HOW λ IS LARGE!

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BUT ACTUAL SOAP FILMS HAVE A LIMITATION ON THEIR SIZE

$\Rightarrow$ 3D OBJECTS WITH THICKNESS OF ORDER $\approx 10^{-6}$ m

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MORE PRECISE EQUATION FROM BALANCE OF PRESSURE WITH THICKNES

(DEFAY-PRIGOGINE - SURFACE & ADSORPTION - 1966)

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MORE PRECISE EQUATION FROM BALANCE OF PRESSURE WITH THICKNES

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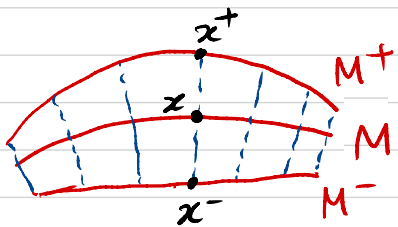
THIS EQUATION IMPLIES THAT THE MIDSECTION OF A "FILM WITH THICKNESS" HAS SMALL (BUT TYPICALLY NON-ZERO) MEAN CURVATURE EVEN WHEN THERE IS NO GRAVITY $\Rightarrow$ WANT TO UNDERSTAND SURFACES WITH SMALL M.C.!!

BALANCE OF PRESSURES

M IDEAL MID-SURFACE $\quad 2h(x)$ THICKNESS AT $x \in M$

$M^+ = \{x + h(x)\nu_M(x) : x \in M\}$ "TOP" INTERFACE

$M^- = \{x - h(x)\nu_M(x) : x \in M\}$ "BOTTOM" INTERFACE



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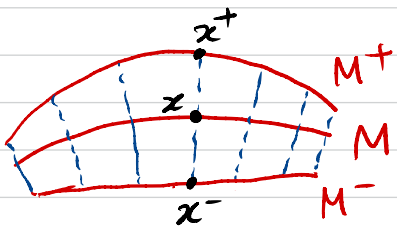
σ SURFACE TENSION



AT $x^+ = x + h(x)\nu_M(x)$

$p(x^+) = p_0 - \sigma H_{M^+}(x^+)$

p_0 ATMOSPHERIC PRESSURE

BALANCE OF PRESSURES

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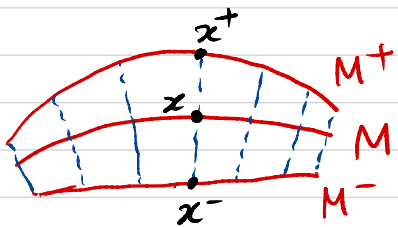

AT $x^+ = x + h(x)\nu_M(x)$

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AT $x^- = x - h(x)\nu_M(x)$

$$p(x^-) = p_0 + \sigma H_{M^-}(x^-)$$

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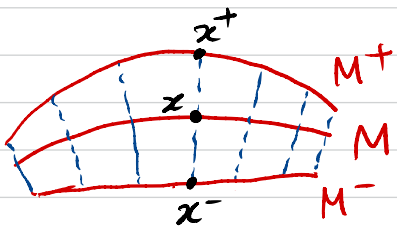

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σ SURFACE TENSION

p_0 ATMOSPHERIC PRESSURE

ρ DENSITY OF FLUID

g EARTH'S GRAVITY

& BECAUSE OF GRAVITY $p(x^-) - p(x^+) = \rho g \, 2h(x)\nu_M(x) \cdot e_3$

BALANCE OF PRESSURES

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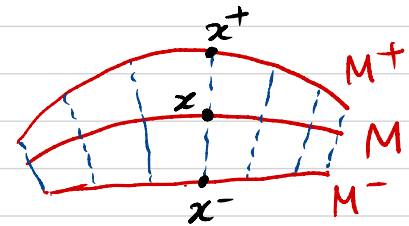

EQUILIBRIUM CONDITION

FOR EVERY $x \in M$

$$\sigma H_{M^-}(x^-) + \sigma H_{M^+}(x^+) = \rho g\, 2h(x)\, \nu_M(x) \cdot e_3$$

σ SURFACE TENSION

p_0 ATMOSPHERIC PRESSURE

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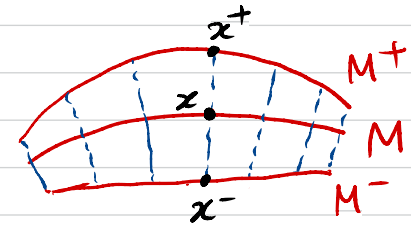

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SANITY CHECK

IF $g = 0$ & $h \equiv 0$ THEN $H_M = 0$ ✓

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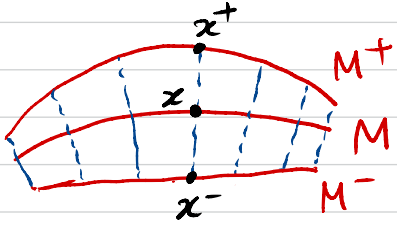

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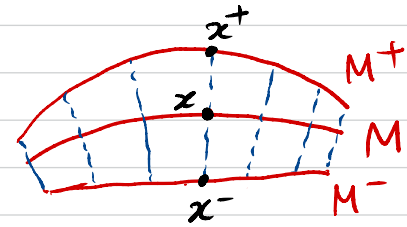

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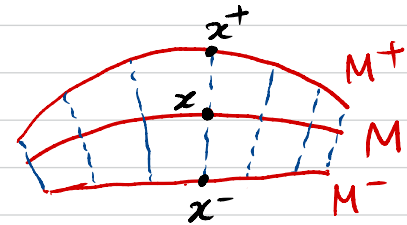

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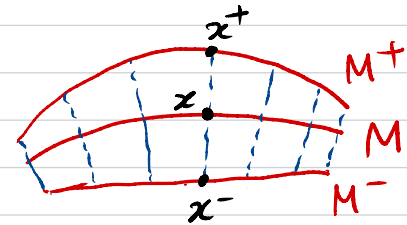

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THEN AT SUCH x: $H_M(x) = O(h)\, \nu_M \cdot e_3$ OR $H_M(x) = O(h^2)\, k_i k_j k_\ell$

$(g \neq 0)$ $(g = 0)$

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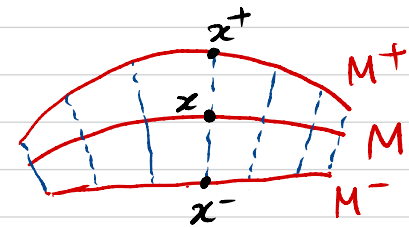

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THEREFORE THE MEAN CURVATURE OF THE MID-SECTION IS SMALL!!

THM (SCARDICCHIO M STUVARD '18) ALMOST-MINIMAL LIPSCHITZ GRAPHS

LET N BE A SMOOTH COMPACT MINIMAL HYPERSURFACE WITH BOUNDARY IN $\mathbb{R}^{n+1}$

WHICH IS STRICTLY STABLE: $\int_N |\nabla^N \varphi|^2 - |A_N|^2 \varphi^2 \geq \lambda \int_N \varphi^2 \quad \forall \varphi \in C_c^\infty(N)$.

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LET $M = \{x + u(x)\nu_N(x) : x \in N\}$, $\|u\|_{C^0(N)} + \mathrm{Lip}(u) \leq \varepsilon_0(N)$.

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THEN $$0 \leq \mathcal{H}^n(M) - \mathcal{H}^n(N) \leq C(N) \int_M |H_M|^2$$

MORALLY: $f(x) = \mathcal{H}^n(M)$, $x_0 = N$, $f'(x_0) = 0$, $f''(x_0) = \lambda > 0$

$$\Rightarrow f'(x)^2 \geq \frac{1}{C}[f(x) - f(x_0)]$$

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$$\|u\|_{C^0(N)} \leq C(N,q) \left\{ \int_M |H_M|^2 + \int_M |H_M|^q \right\} \qquad \forall q > n/2.$$

THM (SCARDICCHIO & STUVARD '18) **ALMOST-MINIMAL LIPSCHITZ GRAPHS**

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THEN

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RMK: SMALLNESS OF $\int_M |H_M|^2$ **DOES NOT** IMPLY GRAPHICALITY BY ALLARD!!
(CRITICAL EXPONENT FOR REGULARITY)

THM (SCARDICCHIO M STUVARD '18)

LET W COMPACT ORIENTABLE $(n-1)$ DIM BOUNDARY IN $\mathbb{R}^{n+1}$

$\{M_j\}_j$ COMPACT ORIENT. n DIM SURFACES IN $\mathbb{R}^{n+1}$

$\partial M_j = f_j(W)$ WITH $f_j \to id$ IN $C^1(W)$

$\sup_j \{\operatorname{diam} M_j, \mathcal{H}^n(M_j)\} < \infty$ $\qquad \lim_{j\to\infty} \int_{M_j} |H_{M_j}| = 0.$

THM (SCARDICCHIO & STUVARD '18)

LET W COMPACT ORIENTABLE $(n-1)$ DIM BOUNDARY IN $\mathbb{R}^{n+1}$

$\{M_j\}_j$ COMPACT ORIENT. n DIM SURFACES IN $\mathbb{R}^{n+1}$

$\partial M_j = f_j(W)$ WITH $f_j \to id$ IN $C^1(W)$

$\sup_j \{\operatorname{diam} M_j, \mathcal{H}^n(M_j)\} < \infty \qquad \lim_{j\to\infty} \int_{M_j} |H_{M_j}| = 0.$

HP1 THE PLATEAU'S PROB OF W IS FINITE & REGULAR AT BOUNDARY

THM (SCARDICCHIO & STUVARD '18)

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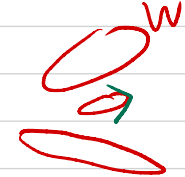
HP2 OK

HP2 NO



HP1 THE PLATEAU'S PROB OF W IS FINITE & REGULAR AT BOUNDARY

HP2 $\forall W'$ CONNECTED COMP OF W & FOR $\mathcal{H}^{n-1}$-A.E. $x \in W'$

$\exists\, \nu_1, \nu_2 \in S^n$, $\nu_1 \cdot \nu_2 < 1$, SUCH THAT $W \subseteq x + \{y : y\cdot\nu_1 \geq 0, y\cdot\nu_2 \geq 0\}$

THEN ...

BUBBLING WHEN HP2 FAILS!

THM (SCARDICCHIO & STUVARD '18) LET W COMPACT ORIENTABLE $(n-1)$ DIM BOUNDARY IN $\mathbb{R}^{n+1}$

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HP1 THE PLATEAU'S PROB OF W IS FINITE & REGULAR AT BOUNDARY

HP2 ACUTE ANGLE CONTAINMENT FOR W

THEN

NO BUBBLING $\exists \begin{cases} \partial N = \Gamma (\subseteq W) \\ N \text{ MINIMAL SURFACE} \end{cases}$ $\exists\, E_j \subseteq \mathbb{R}^{n+1}$ OPEN

S.T. $\mathcal{L}^{n+1}(E_j) + \mathcal{H}^n(\partial E_j \setminus (N \cup M_j)) \to 0 \quad j \to \infty$

THM (SCARDICCHIO M STUVARD '18) LET W COMPACT ORIENTABLE $(n-1)$ DIM BOUNDARY IN $\mathbb{R}^{n+1}$

$\{M_j\}_j$ COMPACT ORIENT. n DIM SURFACES IN $\mathbb{R}^{n+1}$ $\quad \sup_j \{\operatorname{diam} M_j, \mathcal{H}^n(M_j)\} < \infty$

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HP1 THE PLATEAU'S PROB OF W IS FINITE & REGULAR AT BOUNDARY

HP2 ACUTE ANGLE CONTAINMENT FOR W

THEN **NO BUBBLING** $\exists$ N MINIMAL SURFACE $\exists$ $E_j \subseteq \mathbb{R}^{n+1}$ OPEN

S.T. $\mathcal{L}^{n+1}(E_j) + \mathcal{H}^n(\partial E_j \setminus (N \cup M_j)) \to 0 \quad j \to \infty$

STRONG CONVERGENCE IF $\sup_j \int_{M_j} |H_{M_j}|^p < \infty$ FOR SOME $p > n$

THEN M_j LIPSCHITZ GRAPH ON N & ESTIMATES FROM PREVIOUS THM.