THREE LECTURES ON ISOPERIMETRIC & PLATEAU-TYPE PROBLEMS

FRANCESCO MAGGI

UNIVERSITY OF TEXAS AT AUSTIN

GEOMETRIC ANALYSIS & PDE ON GARDA LAKE

8-10 JUNE 2022

LECTURE 3

CAPILLARITY THEORY MEETS WITH PLATEAU'S PROBLEM

A LIST OF PHYSICAL INADEQUACIES OF PLATEAU'S PROBLEM

(AS A MODEL FOR SOAP FILMS)

PLATEAU'S PROBLEM GIVEN W WITH $\partial W = 0$

FIND M SUCH THAT $H_M = 0$ & $\partial M = W$.

ONE LACK OF A LENGTH SCALE (SEE LECTURE 2, ALMOST MINIMAL SURFACES)

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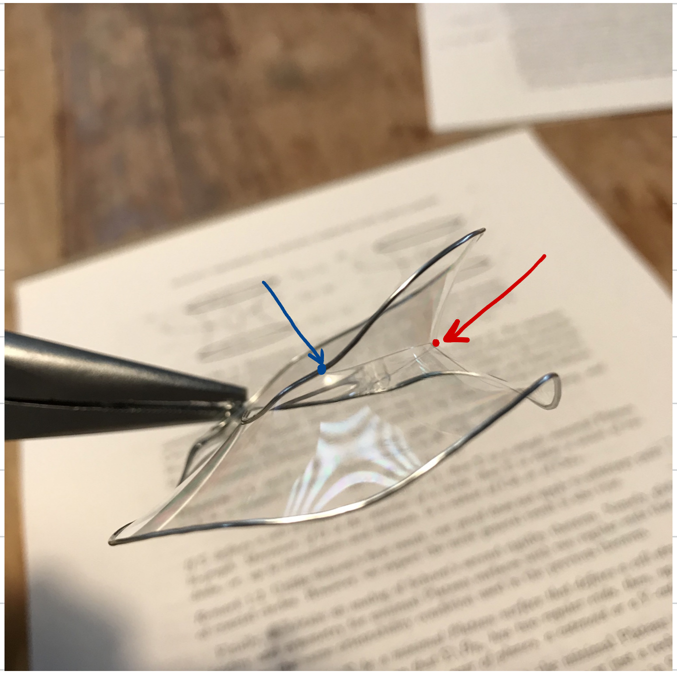
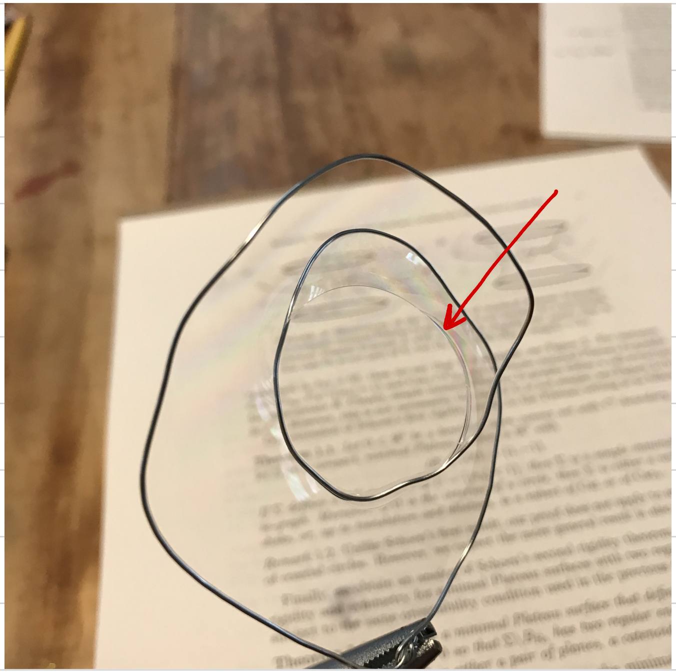
THREE NON-UNIQUENESS & INSTABILTY

TOO MUCH REGULARITY P.P.: FIND M S.T. $\partial M = W$, $H_M = 0$

MISMATCH BETWEEN THEORY (SMOOTH SURFACES)

& EXPERIMENTS (PLATEAU, CANONICAL SINGULARITIES)





• = BOUNDARY SINGULARITY • = INTERIOR SINGULARITY

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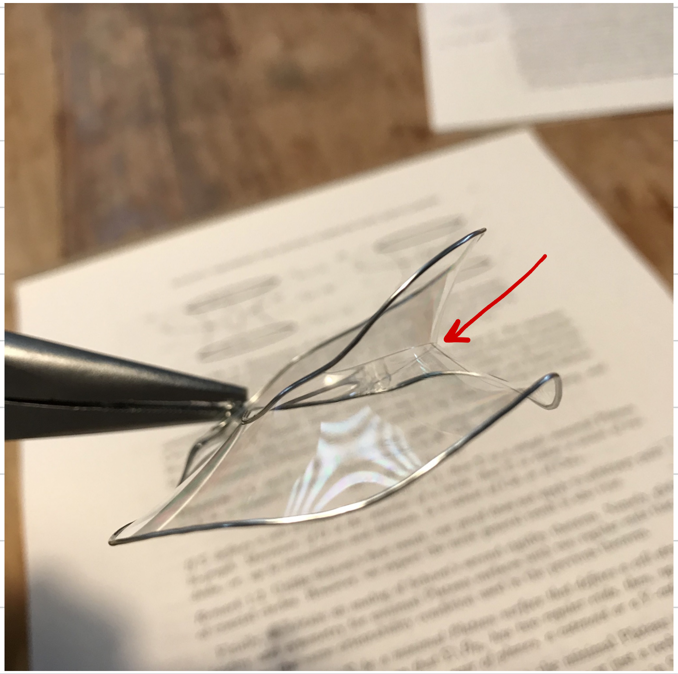
PLATEAU'S LAWS SOAP FILMS CORRESPOND



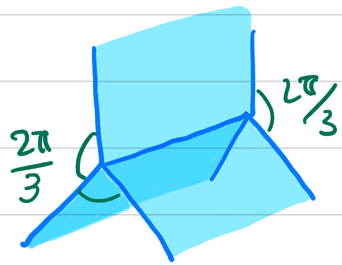
TO MINIMAL SURFACES JOINED ALONG

LINES OF Y-POINTS

WHICH ARE EITHER CLOSED OR

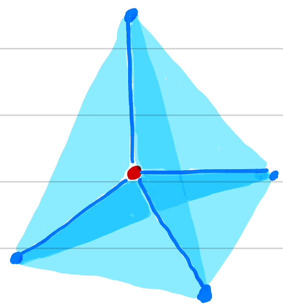
END UP INTO T-POINTS.

Y =



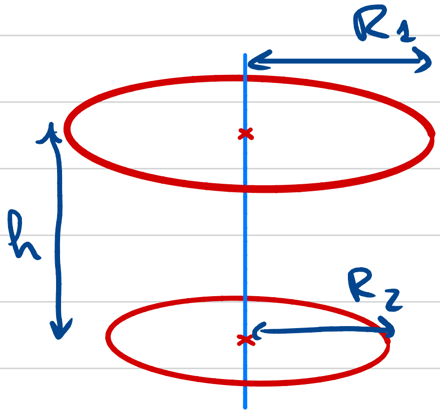
3 HALF PLANES

T =



6 PLANAR SECTORS

NON UNIQUENESS: $\exists$ 3 MINIMAL SURFACES IN $\mathbb{R}^3$ SPANNING TWO PARALLEL CIRCLES (SCHOEN 1983)



TWO CIRCLES

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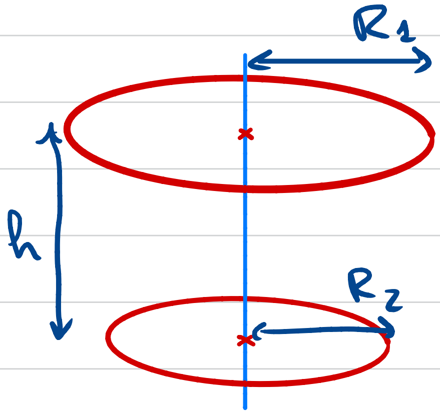
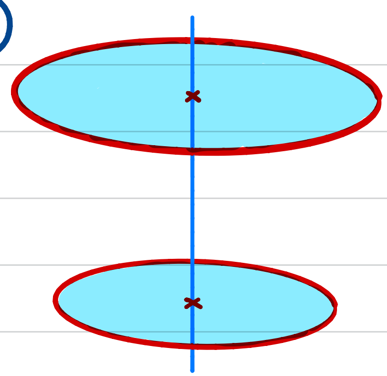


TWO CIRCLES

$\Longrightarrow$

1



TWO DISKS

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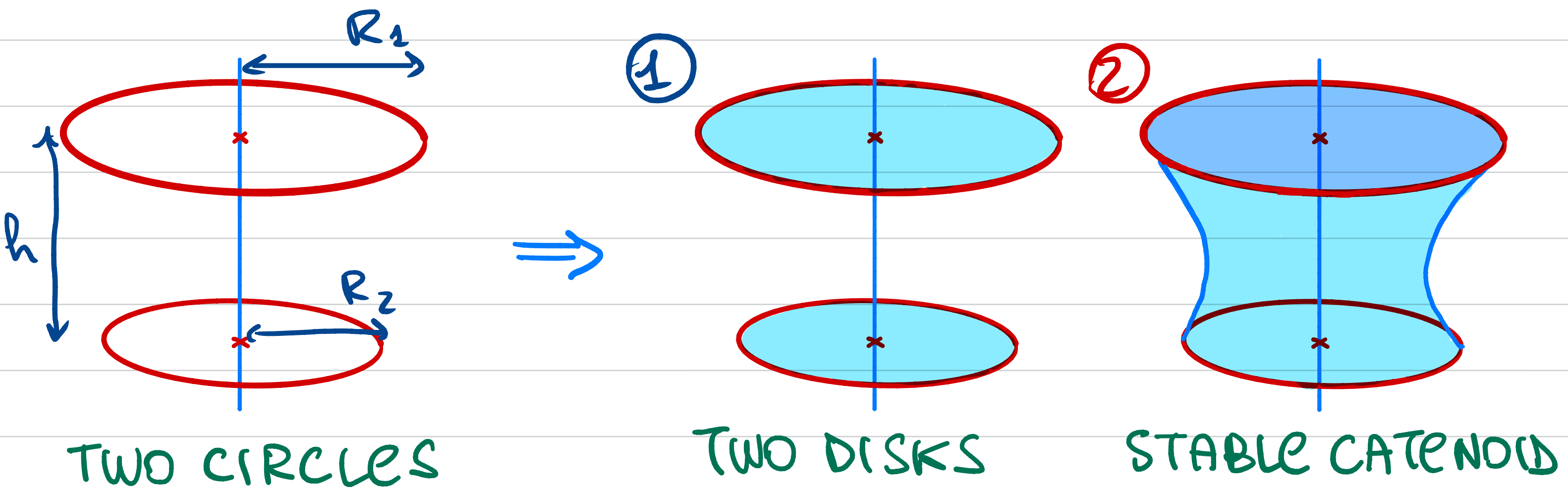

POSITIVE SECOND VAR.

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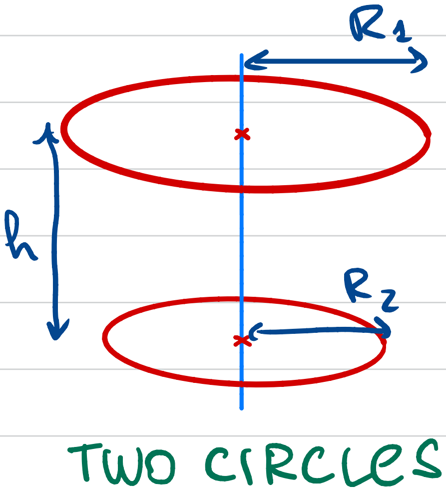
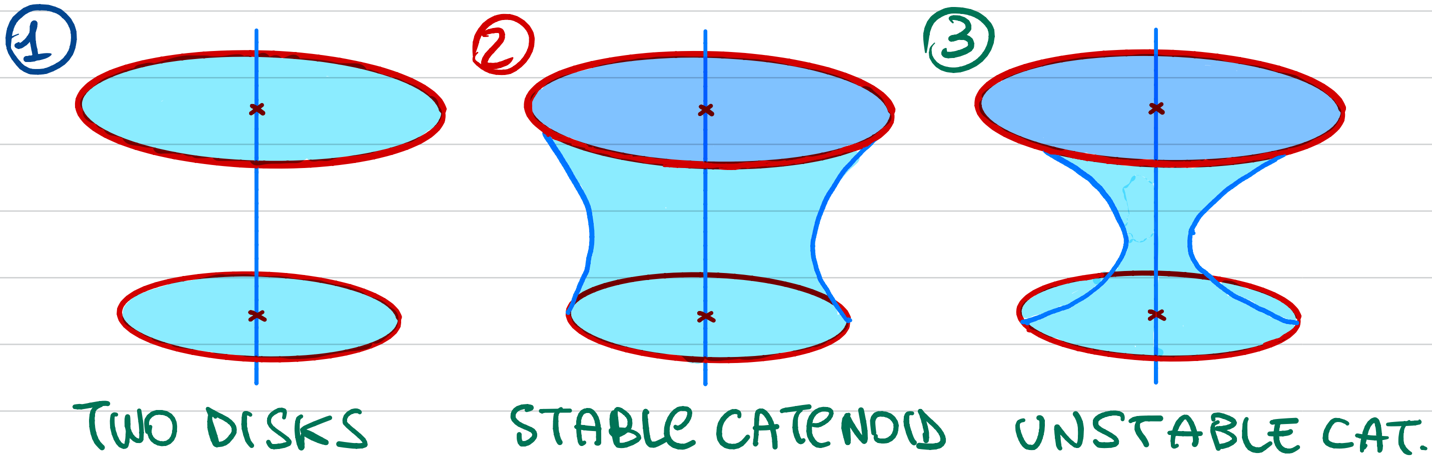


$\Longrightarrow$



POSITIVE SECOND VAR.

NEGATIVE SECOND VAR.
UNPHYSICAL

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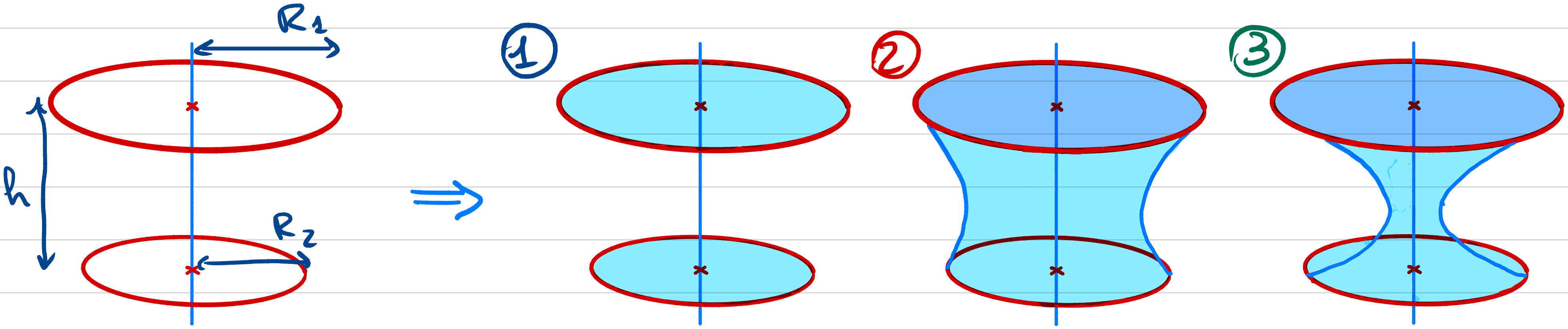
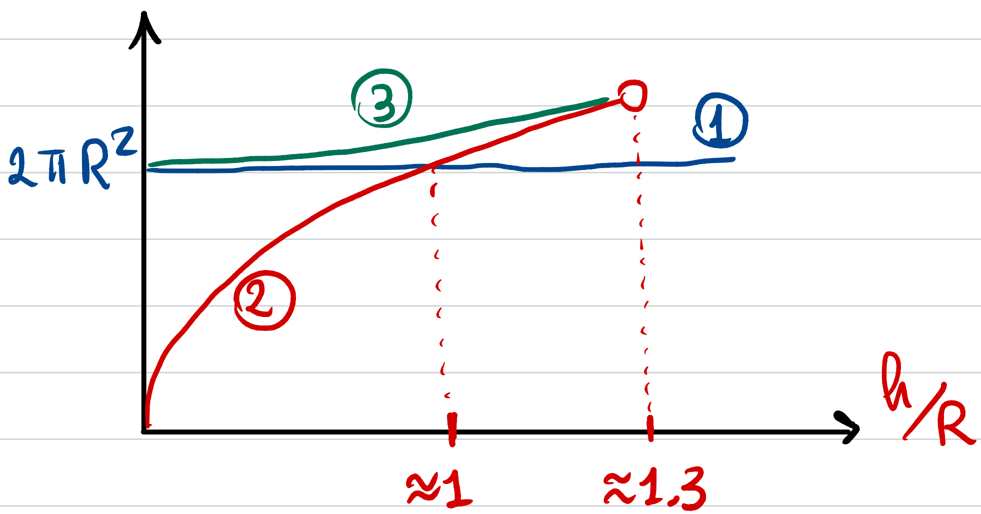


TWO CIRCLES | TWO DISKS | STABLE CATENOID | UNSTABLE CAT.

POSITIVE SECOND VAR. | NEGATIVE SECOND VAR. UNPHYSICAL

$R_1 = R_2$



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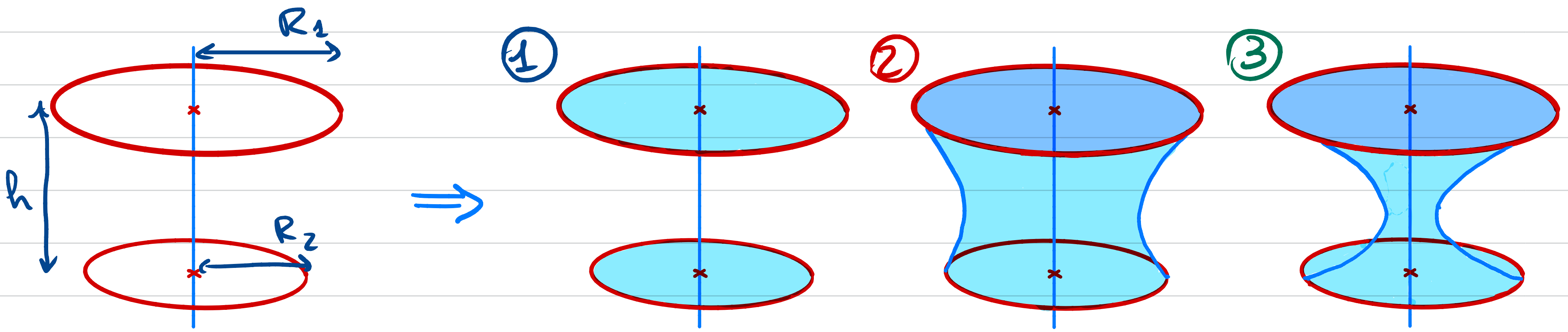
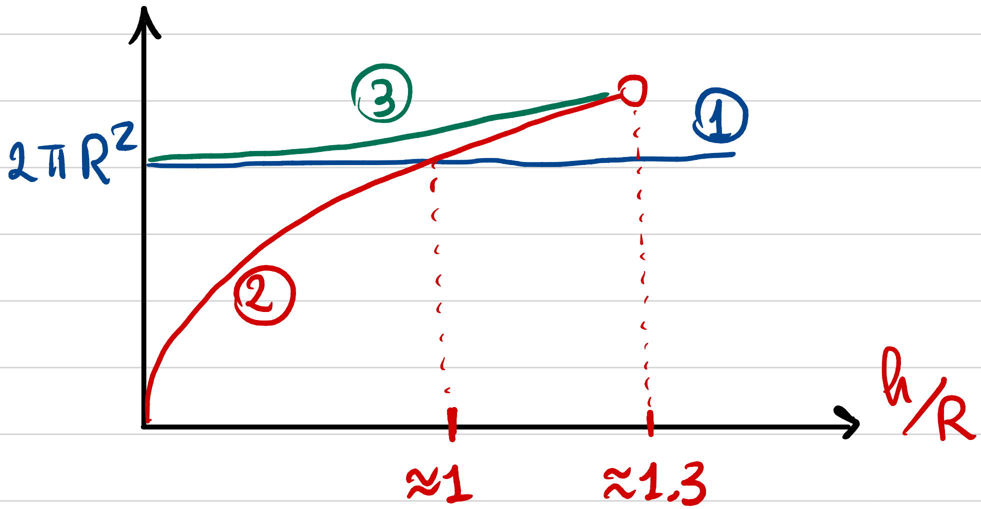


TWO CIRCLES — TWO DISKS — STABLE CATENOID — UNSTABLE CAT.

STABLE CATENOID: POSITIVE SECOND VAR.

UNSTABLE CAT.: NEGATIVE SECOND VAR. UNPHYSICAL

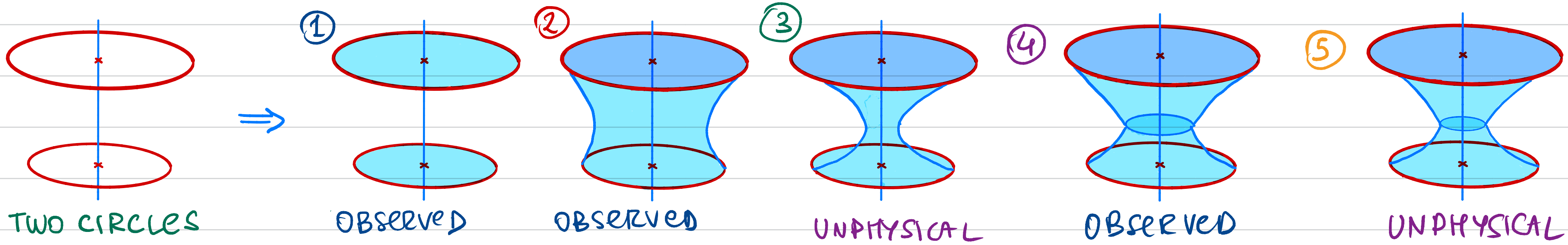
$R_1 = R_2$



NON-UNIQUENESS BETWEEN 1 & 2 IS PHYSICAL & DEPENDS ON THE FILM-GENERATING PROCESS

NON UNIQUENESS: $\exists$ 3 MINIMAL SURFACES IN $\mathbb{R}^3$ SPANNING TWO PARALLEL CIRCLES (SCHOEN 1983)

$\exists$ 5 IF PLATEAU SINGULARITIES ARE ALLOWED $\rightarrow$ SINGULAR CATENOIDS:
(J. BERNSTEIN & M. 2020)



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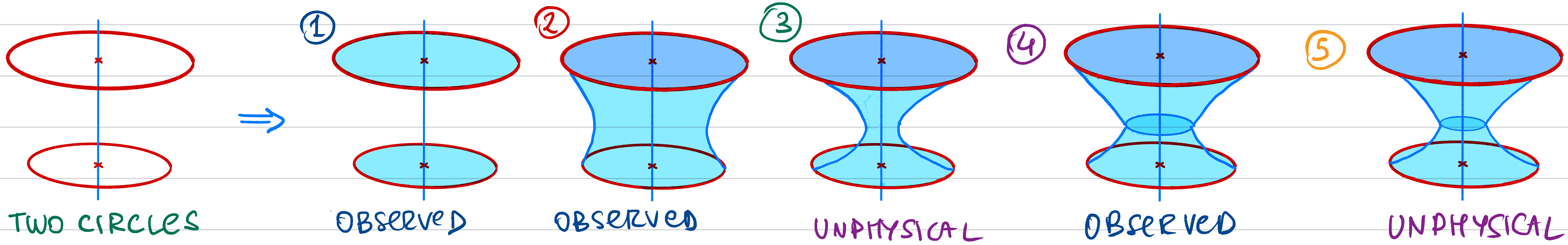

QUESTION HOW CAN WE RELATE 4 TO AREA MINIMIZATION?

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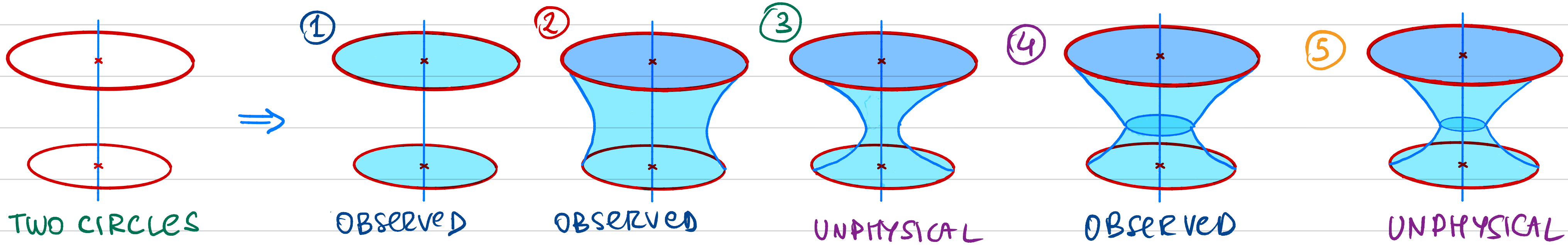

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REMARK PARAMETRIZED MIN. SURF. (DOUGLAS RADO), SETS OF FINITE PERIMETER, AREA MINIMIZING CURRENTS... IN $\mathbb{R}^3$ MINIMIZERS ARE SMOOTH $\Rightarrow$ ONE CAN ONLY OBSERVE 1 & 2 AS MINIMIZERS

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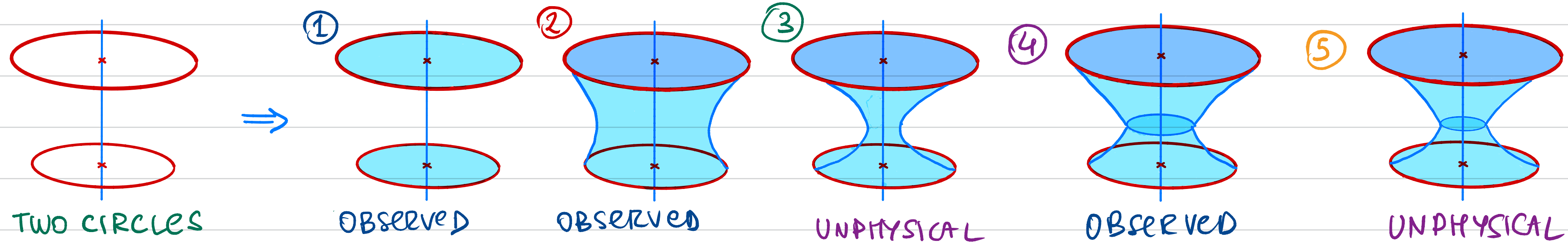

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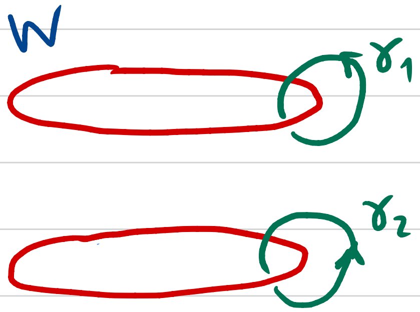
NON-SMOOTH APPROACH ALMGREN, DAVID, HARRISON-PUGH...

HARRISON-PUGH APPROACH DATA WIRE FRAME: $W \subseteq \mathbb{R}^{n+1}$ COMPACT SET

SPANNING CLASS $\mathcal{C}$ FAMILY OF EMBEDD. OF S^1 IN Ω CLOSED BY Ω-HOMOTOPIES



$\mathcal{C} = \{[\gamma_1], [\gamma_2]\}$

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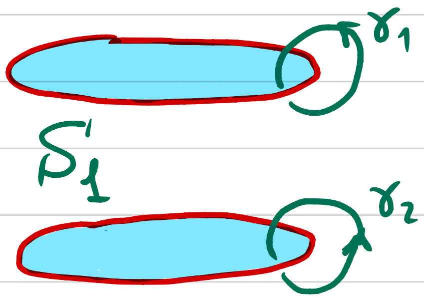
DEFINITION A COMPACT SET S is **$\mathcal{C}$-SPANNING W**

IF $S \cap \gamma \neq \emptyset \quad \forall \gamma \in \mathcal{C}$



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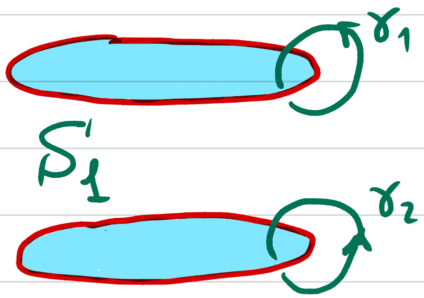

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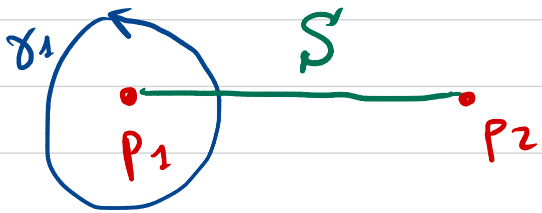
EXAMPLE 1 $n=1$, $W = \{p_1, p_2\}$

$\mathcal{C} = \{ [\gamma_1] \}$



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EXAMPLE 1 $n=1$, $W = \{p_1, p_2\}$ **EX 2** $n=1$, $W=\{p_1,p_2,p_3\}$, $\mathcal{C}=\{[\gamma_1],[\gamma_2],[\gamma_3]\}$

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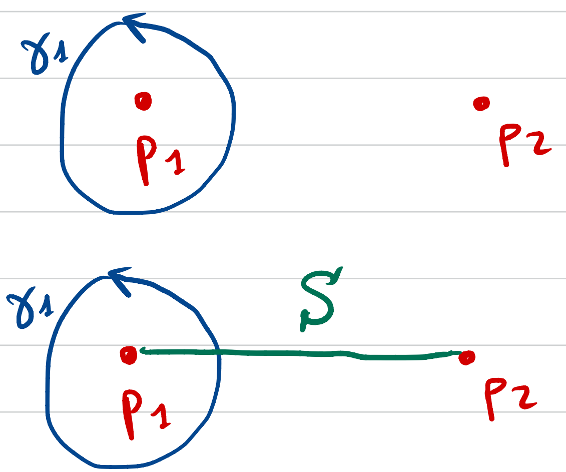

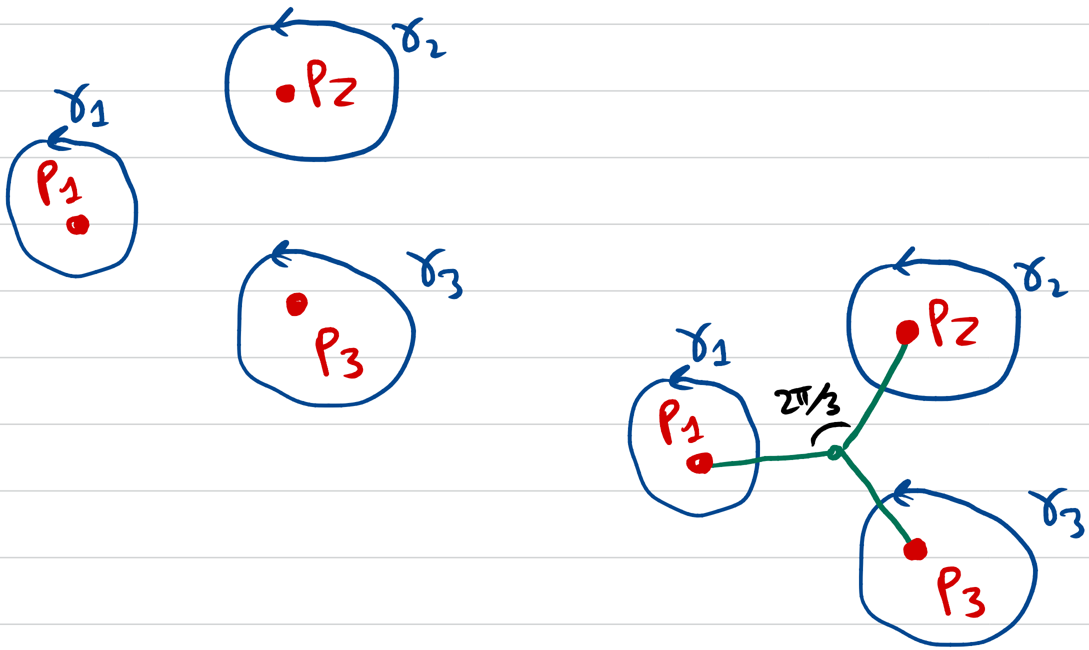

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EX 3 $n=1$, $W = \{p_i\}_{i=1}^4$

$\mathcal{C} = \{ [\gamma_i] \}_{i=1}^2$

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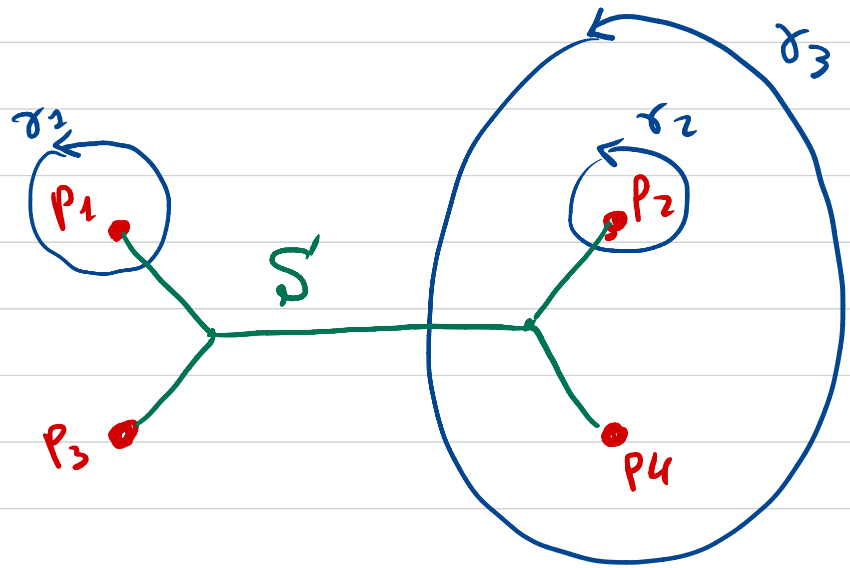

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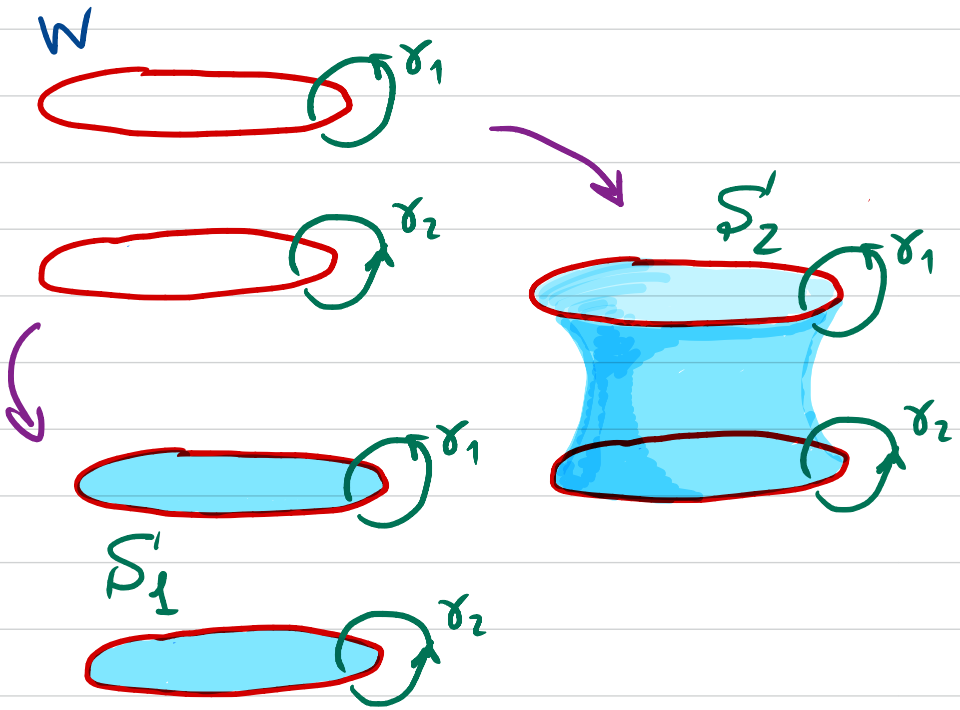
EX 5 $W = 2$ CIRCLES $\mathcal{C} = \{[\gamma_i]\}_{i=1}^2$



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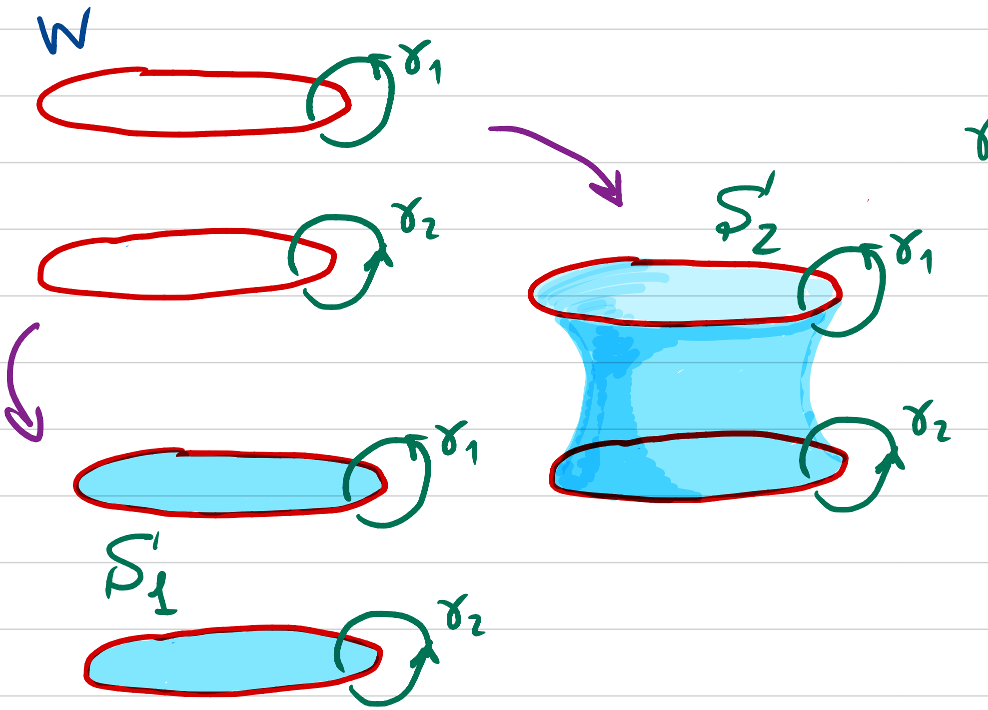

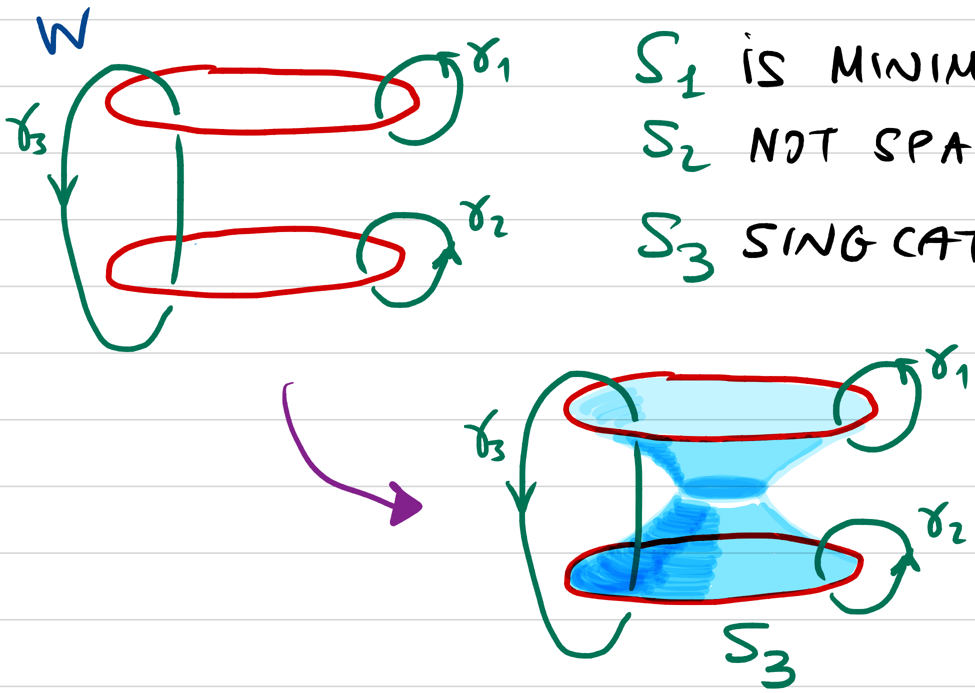

S_1 IS MINIMIZER
S_2 NOT SPANNING
S_3 SING CATENOID

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THM (HARRISON-PUGH '13, '16 / DE LELLIS, GHIRALDIN, M. '14)

IF W COMPACT & $\ell < \infty$ THEN $\exists\, S$ MINIMIZER OF ℓ

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MOREOVER S IS "ALMGREN-MINIMIZING IN $\Omega = \mathbb{R}^{n+1} \setminus W$" THAT IS

$$\mathcal{H}^n(S) \leq \mathcal{H}^n(f(S))$$

$\forall f : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ LIPSCHITZ WITH $\{f \neq id\} \subset\subset \Omega$

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MUCH STRONGER THAN "DIFFEOMORPHISM"

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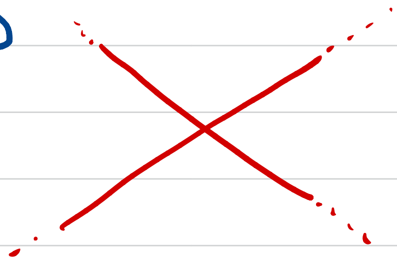
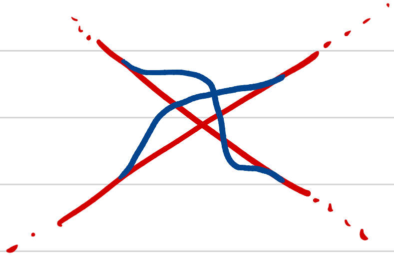
EXAMPLE $n=1$

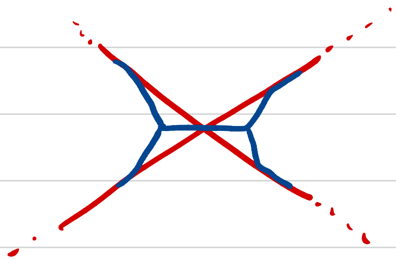
SATISFIES $(\star)$ WITH f DIFFEOMORPHISM BUT NOT WITH f LIPSCHITZ

S





DIFFEOS ALWAYS PAY MORE



LENGHT IMPROVING DEFORMATION

H.-P. PLATEAU'S PROBLEM $\ell = \inf\{\mathcal{H}^n(S) : S$ IS $\mathcal{C}$-SPANNING $W\}$

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$\mathcal{H}^n(S) \le \mathcal{H}^n(f(S))$ $\forall f : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ LIPSCHITZ WITH $\{f \neq id\} \subset\subset \Omega$

ALMGREN 76 IF S IS ALMGREN-MINIMIZING IN Ω

THEN $\exists\, \Sigma \subseteq S$ CLOSED SUCH THAT $H_S = 0$ ON $S \setminus \Sigma$ & $\mathcal{H}^n(\Sigma) = 0$

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$\mathcal{H}^n(S) \le \mathcal{H}^n(f(S))$ $\forall f : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ LIPSCHITZ WITH $\{f \neq id\} \subset\subset \Omega$

ALMGREN 76 IF S IS "ALMGREN-MINIMIZING IN $\Omega = \mathbb{R}^{n+1} \setminus W$"

THEN $\exists\, \Sigma \subseteq S$ CLOSED SUCH THAT $H_S = 0$ ON $S \setminus \Sigma$ & $\mathcal{H}^n(\Sigma) = 0$

TAYLOR 76 MOREOVER, IF $n = 2$, THEN $\forall x \in \Sigma\ \exists\, \rho_x > 0$ SUCH THAT

$S \cap B_{\rho_x}(x)$ IS $C^{1,\alpha}$ DIFFEOMORPHIC TO $Y \cap B_1$ OR $T \cap B_1$

WITH DIFFEO $f : B_{\rho_x}(x) \to B_1(0)$ S.T. $f(x) = 0$, $\nabla f(x) \in O(3)$.

THIS ANSWERS TO "TOO MUCH REGULARITY"

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BUT STILL LACKS OF LENGHT SCALE !

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LECTURE 2 USING BALANCE OF PRESSURES WE FOUND THAT THE PDE

FOR THE MIDSECTION M OF A FILM WITH THICKNESS $2h$ IS

(☆) $H_{M^-}(x^-) + H_{M^+}(x^+) = 0$ ON M

$M^\pm = \{ x^\pm : x \in M \}$ $\quad x^\pm = x \pm h(x)\, \nu_M(x) \quad (x \in M)$

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HOWEVER THICKNESS IS NOT REALLY A FUNDAMENTAL PHYSICAL

PROPERTY OF SOAP FILM → VOLUME IS MORE ROBUST

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HOWEVER THICKNESS IS NOT REALLY A FUNDAMENTAL PHYSICAL

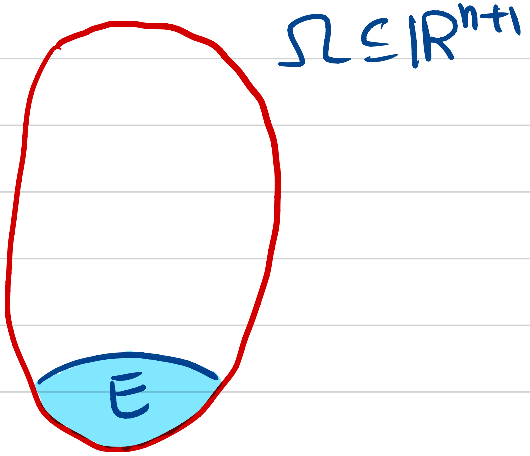
PROPERTY OF SOAP FILM → VOLUME IS MORE ROBUST

ALSO WHAT IS THE ENERGY OF WHICH $(\star)$ IS GOING TO BE

THE EULER LAGRANGE EQUATION ? CAPILLARITY THEORY

SOAP FILMS FROM CAPILLARITY THEORY

LIQUID AT EQUILIBRIUM OCCUPYING REGION $E \subseteq \Omega$ CONTAINER



$$\Phi_\Omega(v) = \inf \{ P(E;\Omega) : E \subseteq \Omega,\ |E| = v \}$$

[RMK: $\Phi_\Omega = \psi_{\mathbb{R}^{n+1};\Omega}$ FROM LECT. 1]

SOAP FILMS FROM CAPILLARITY THEORY

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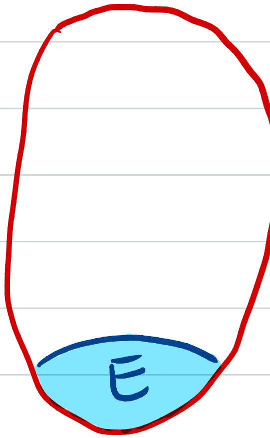

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$E \approx$ HALF BALL OF VOLUME v CONCENTRATING NEAR A POINT OF MAX H_Ω ON $\partial\Omega$ AS $v \to 0^+$

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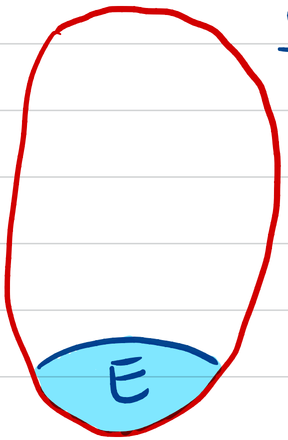

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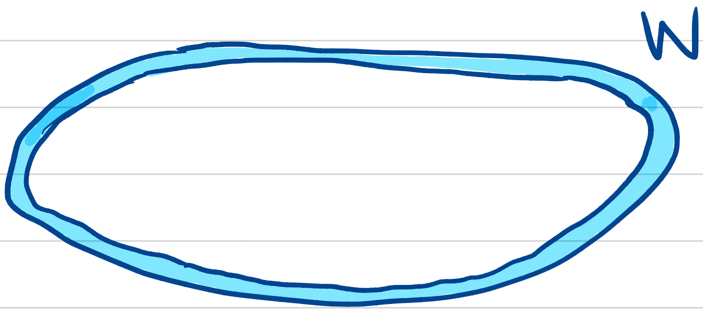
IN THE CASE OF SOAP FILMS

$\Omega = \mathbb{R}^{n+1} \setminus W$, $W = I_\delta(\Gamma)$, Γ CURVE:



SOAP FILMS FROM CAPILLARITY THEORY

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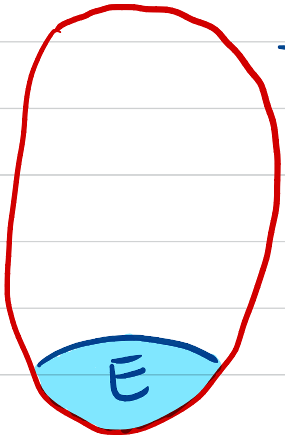

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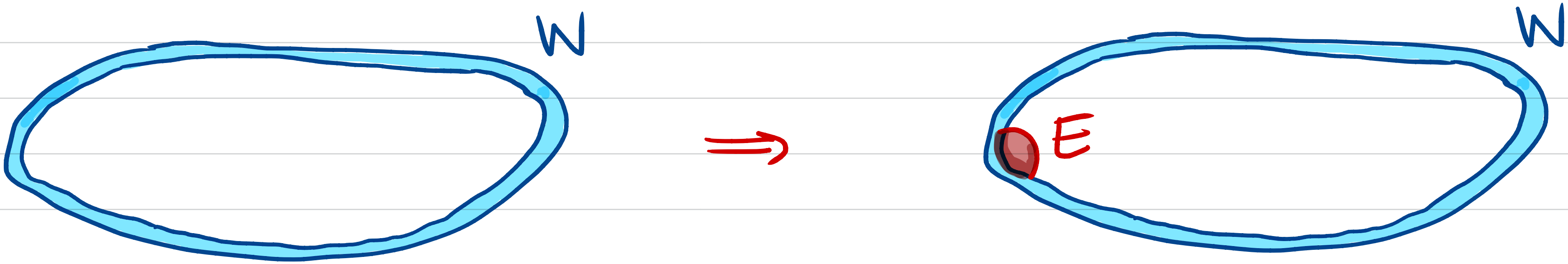

DOESN'T LOOK LIKE A SOAP FILM!

SOAP FILMS FROM CAPILLARITY THEORY WITH HOMOTOPIC SPANNING

PROPOSED IN SCARDICCHIO M. STUVARD '18
& ANALYZED IN KING M. STUVARD '19 '20 '20

GIVEN $W \subseteq \mathbb{R}^{n+1}$ COMPACT,

A SPANNING CLASS $\mathcal{C}$ FOR W, SET

$$\psi_W^{\mathcal{C}}(v) = \inf \{ P(E;\Omega) : |E| = v, E \subseteq \Omega; \; \Omega \cap \partial E \text{ is } \mathcal{C} \text{ spanning } W \}$$

WHERE $\Omega = \mathbb{R}^{n+1} \setminus W$

SOAP FILMS FROM CAPILLARITY THEORY WITH HOMOTOPIC SPANNING

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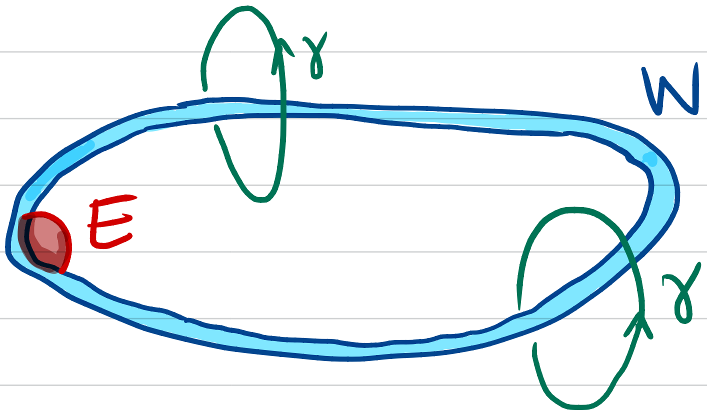
A ROUND DROPLET

IS NOT $\mathcal{C}$ SPANNING W !!!



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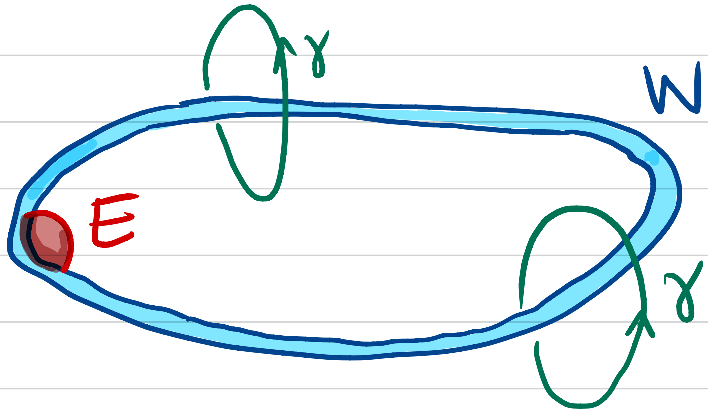
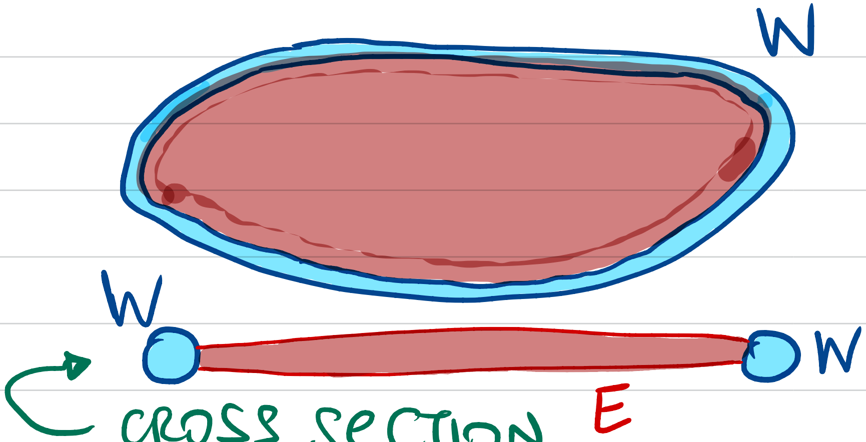


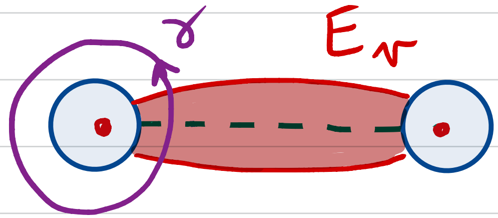
REGIONS OCCUPIED BY LIQUID NEED TO STRETCH ACROSS THE HOLE:



EX 1 W TWO DISKS IN $\mathbb{R}^2$, $\mathcal{C} = \{[\gamma]\}$

S MINIMIZER OF ℓ :

E_v MINIMIZER OF $\Psi_W^{\mathcal{C}}(v)$, v SMALL:



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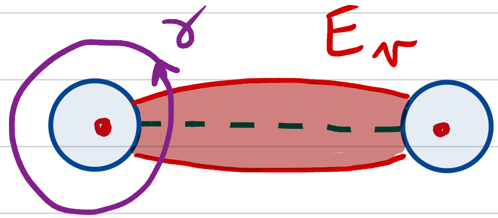

RMK: TWO ARCS OF CURVATURE $H_{E_v} = O(v)$

$\Rightarrow$ ALMOST-MINIMAL SURFACES

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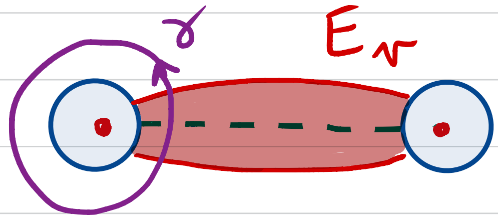

RMK: TWO ARCS OF CURVATURE $H_{E_v} = O(v)$

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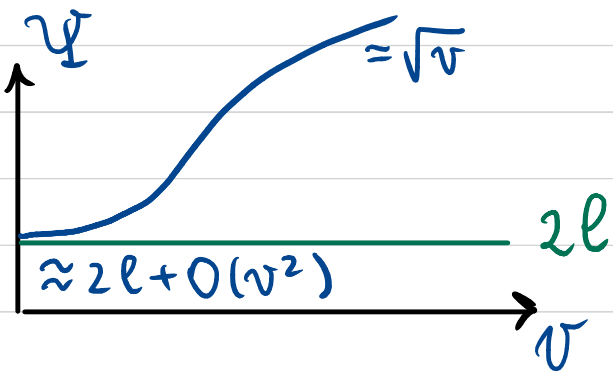
RMK: "$H_{E_v} = \Psi'(v)$"

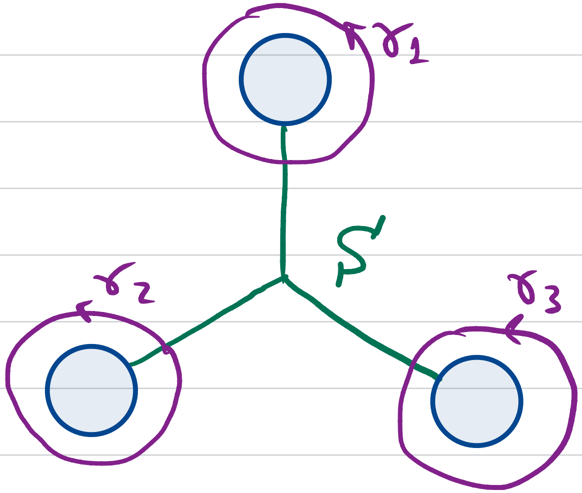
SO $\Psi(v) \approx 2\ell + O(v^2) \to 2\ell$

AS $v \to 0^+$



EX 2 W THREE DISKS IN $\mathbb{R}^2$, $\mathcal{C} = \{[\gamma_1], [\gamma_2], [\gamma_3]\}$

S' MINIMIZER OF ℓ



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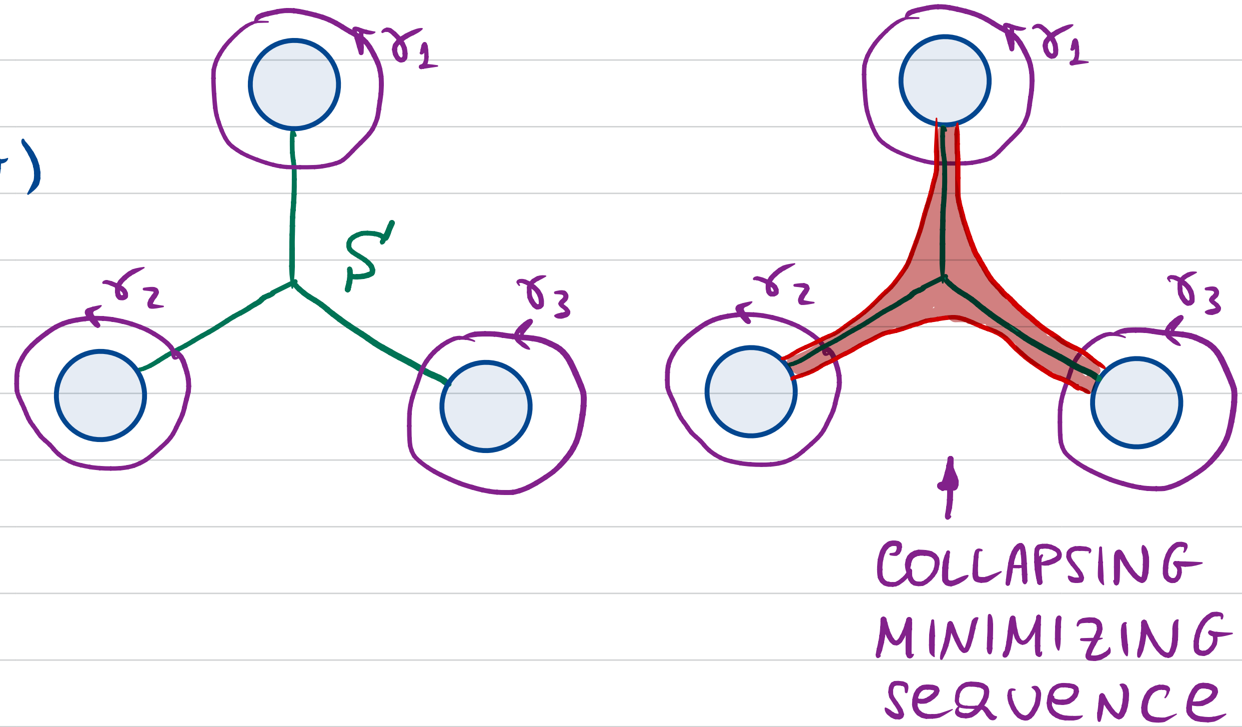
S' minimizer of ℓ

& E_v minimizer of $\Psi_W^{\mathcal{C}}(v)$



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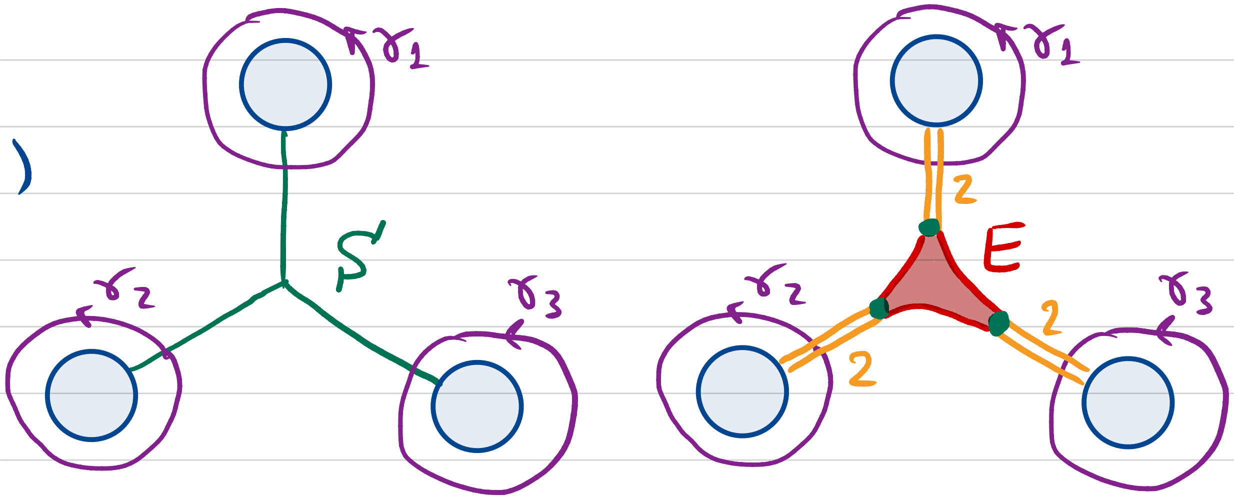

$\Rightarrow E_v$ IS BOUNDED BY

3 CIRCULAR ARCS WITH

NEGATIVE CURVATURE $H_{E_v} = -c/\sqrt{v}$

JOINED TO 3 SEGMENTS WITH MULTIPLICITY 2

AT 3 "FREE BOUNDARY" POINTS

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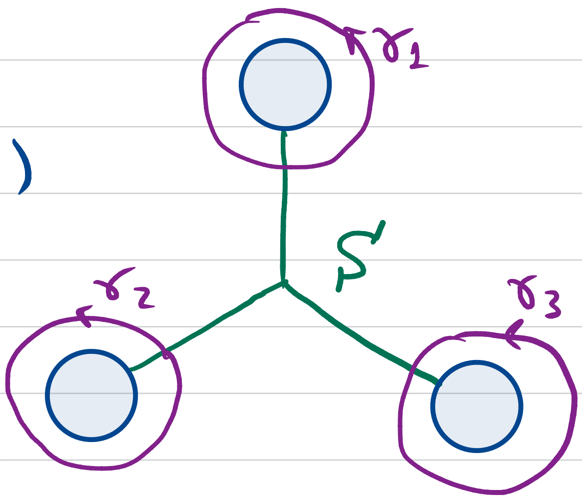

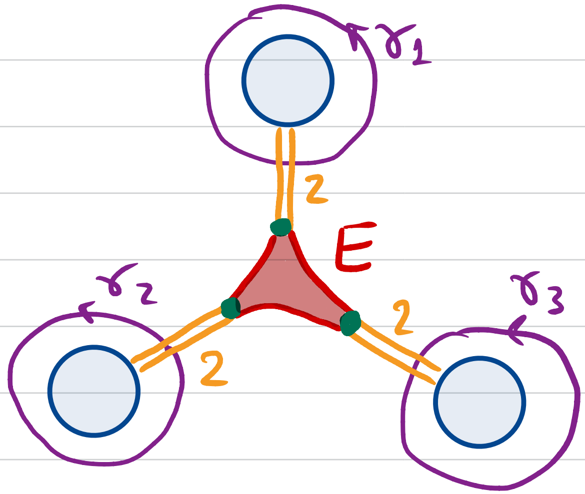

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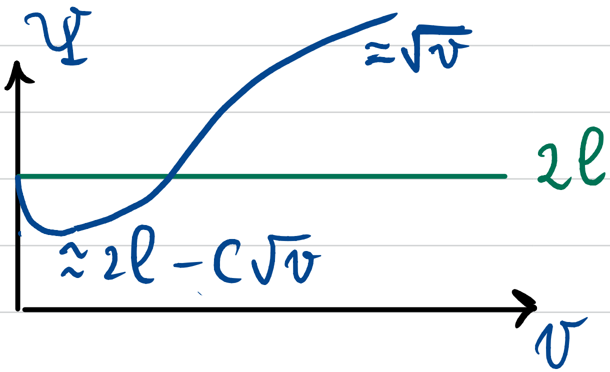

THM (KING M. STUWARD '19) LET W COMPACT WITH SPANNING CLASS $\mathcal{C}$

LET $\ell < \infty$, ∂W SMOOTH, & $\mathbb{R}^{n+1} \setminus I_\tau(W)$ CONNECTED $\forall \tau < \tau_0$

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THEN $\forall$ MINIMIZING SEQUENCE $\{E_j\}_j$ OF $\Psi_W^{\mathcal{C}}(v)$

THERE ARE K A COMPACT SET IN $\Omega=\mathbb{R}^{n+1}\setminus W$, $\mathcal{C}$-SPANNING W

$E\subseteq\Omega$ WITH $|E|=v$, $\Omega\cap\partial E\subseteq K$

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$$P(E_j;\Omega) \to 2\mathcal{H}^n(K \setminus \partial^* E) + \underbrace{\mathcal{H}^n(\Omega \cap \partial^* E)}_{= P(E;\Omega)} = \Psi_W^{\mathcal{C}}(v)$$

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COLLAPSED REGION [CONTINUES]

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MOREOVER $\Psi_W^{\mathcal{C}}$ IS LOWERSEMICONTINUOUS WITH $\Psi(v) \leq 2\ell + C v^{\frac{n}{n+1}}$

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$\exists\ S$ MINIMIZER OF ℓ SUCH THAT

$$2\mathcal{H}^n \llcorner (K \setminus \partial^* E_j) + \mathcal{H}^n \llcorner \partial^* E_j \xrightarrow{*} 2\mathcal{H}^n \llcorner S \quad \text{AS } j \to \infty.$$

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CONJECTURE 1 IF EVERY MINIMIZER OF ℓ IS SMOOTH

THEN NO COLLAPSING

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CONJECTURE 1 IF EVERY MINIMIZER OF ℓ IS SMOOTH

THEN NO COLLAPSING

REMARK IN (★) WE SELECT MINIMIZERS OF ℓ BY

1) MORE SINGULAR
2) MORE CURVATURE

THM (KING M. STUVARD '19) IF (K,E) GENERALIZED MINIMIZER OF $\Psi_W^{e}(v)$

& $\mathcal{E}(K,E) = \underbrace{\mathcal{H}^n(\Omega \cap \partial^* E)}_{P(E;\Omega)} + 2\,\mathcal{H}^n(K \setminus \partial^* E)$

THM (KING M. STUVARD '19) IF (K,E) GENERALIZED MINIMIZER OF $\Psi_W^e(v)$

& $\mathcal{E}(K,E) = \mathcal{H}^n(\Omega \cap \partial^* E) + 2\mathcal{H}^n(K \setminus \partial^* E)$

THEN $\mathcal{E}(K,E) \le \mathcal{E}(f(K), f(E)) \quad \forall f : \Omega \to \Omega \ C^1$-DIFFEOMORPHISM

$$f_t(x) = x + tX(x) + O(t^2)$$

$$\int_{\partial^* E} dw^k X + 2 \int_{K \setminus \partial^* E} du^k X = \lambda \int_{\partial^* E} X \cdot \nu_E$$

$$V = \mathrm{Var}(K, \theta) \qquad \theta = \begin{cases} 2 & \text{on } K \setminus \partial^* E \\ 1 & \text{on } \partial^* E \end{cases}$$

$$\delta V(X) = \int \vec{H} \cdot X \, dV$$

$$\boxed{\vec{H} = \begin{cases} \lambda & \partial^* E \\ 0 & K \setminus \partial^* E \end{cases}}$$

THM (KING M. STUWARD '19) IF (K,E) GENERALIZED MINIMIZER OF $\Psi^{e}_{W}(v)$

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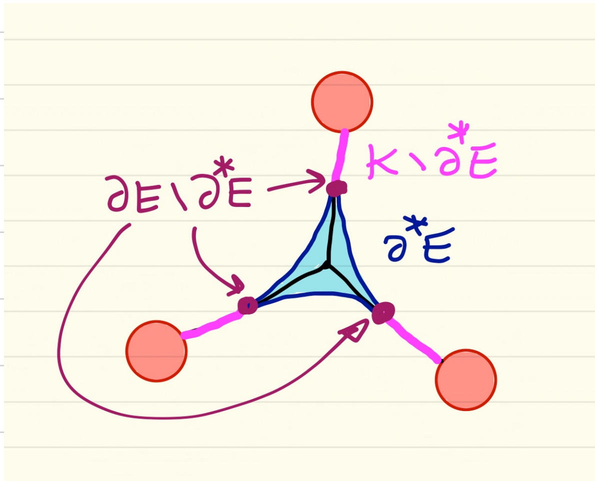
THEN $\mathcal{E}(K,E) \le \mathcal{E}(f(K),f(E))$ $\quad\forall f:\Omega\to\Omega$ C^1-**DIFFEOMORPHISM**

S.T. $|f(E)| = |E|$

MOREOVER BY ALLARD

$\exists\, \Sigma \subseteq K$ CLOSED S.T.

$K\setminus(\Sigma\cup\partial E)$ SMOOTH MINIMAL SURFACE



THM (KING M. STUVARD '19) IF (K,E) GENERALIZED MINIMIZER OF $\Psi^{e}_{W}(v)$

& $\mathcal{E}(K,E) = \mathcal{H}^n(\Omega \cap \partial^* E) + 2\,\mathcal{H}^n(K \setminus \partial^* E)$

THEN $\mathcal{E}(K,E) \le \mathcal{E}(f(K), f(E))$ $\quad \forall f : \Omega \to \Omega$ C^1-DIFFEOMORPHISM

S.T. $|f(E)| = |E|$

MOREOVER BY ALLARD

$\exists\, \lambda \in \mathbb{R}$ $\exists\, \Sigma \subseteq K$ CLOSED S.T.

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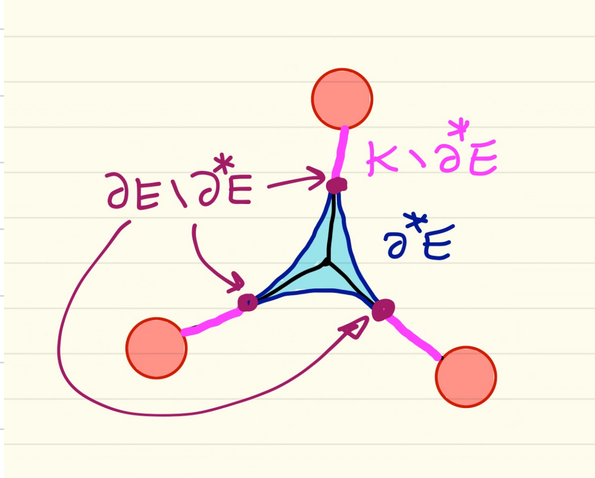
$\Omega \cap \partial^* E$ SMOOTH λ-CMC SURFACE



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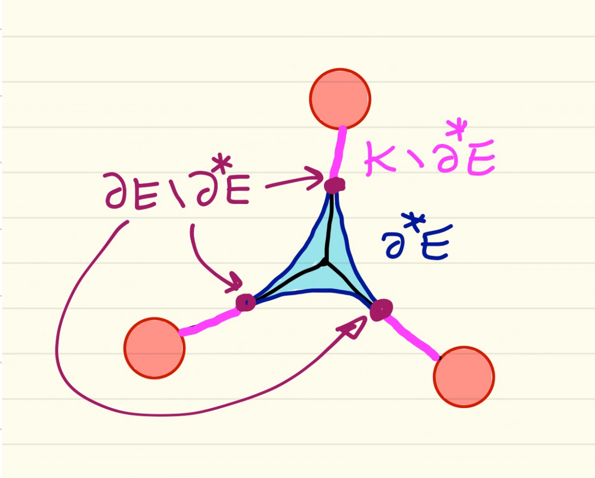

$\Omega \cap \partial^* E$ SMOOTH λ-CMC SURFACE

$\mathcal{H}^n(\Sigma \setminus \partial E) = 0$

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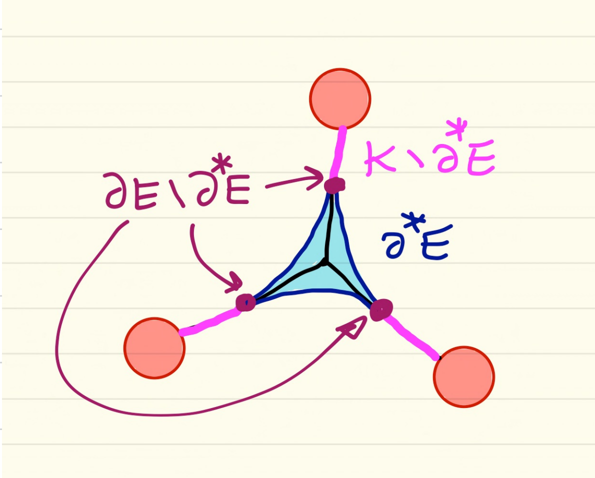
$\Omega \cap \partial^* E$ SMOOTH λ-CMC SURFACE

$\mathcal{H}^n(\Sigma \setminus \partial E) = 0$

$\partial E \setminus \partial^* E$ HAS EMPTY INTERIOR



INVESTIGATING THE COLLAPSED REGION

IN PRINCIPLE $K \setminus \partial E$ COULD CONSIST OF AN "EXTERIOR" & AN "INTERIOR"

EXTERIOR : $K \setminus \overline{E}$ RMK E IS OPEN

INTERIOR : $K \cap E$

INVESTIGATING THE COLLAPSED REGION

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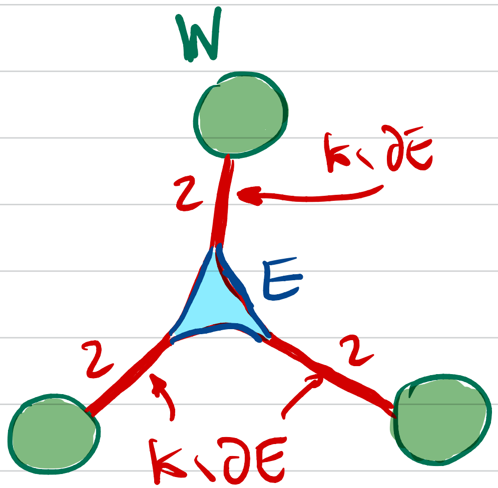

$K \setminus \partial E = K \setminus \overline{E}$

EXTERIOR COLLAPSING

INVESTIGATING THE COLLAPSED REGION

IN PRINCIPLE $K \setminus \partial E$ COULD CONSIST OF AN "EXTERIOR" & AN "INTERIOR"

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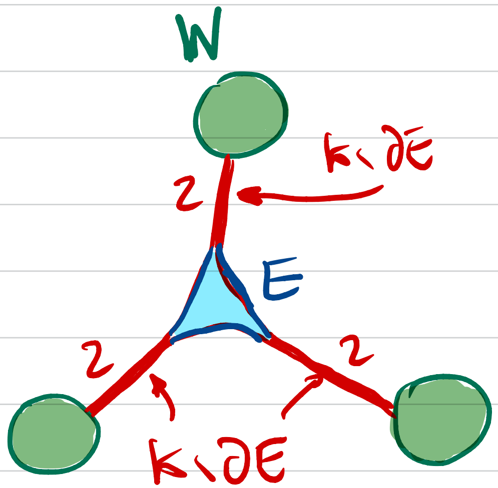

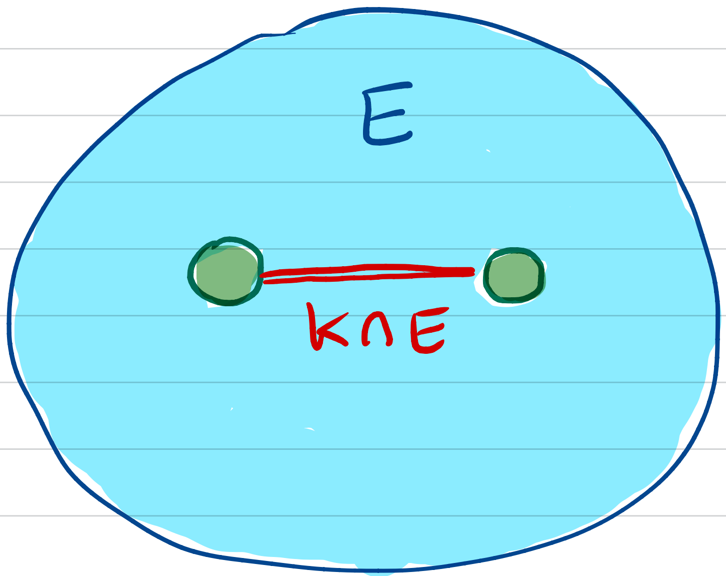

$K \setminus \partial E = K \setminus \bar{E}$

EXTERIOR COLLAPSING

$K \setminus \partial E = K \cap E$

INTERIOR COLLAPSING

(NOT SURE IF IT EVER OCCURS!)

THM (KING M. STUVARD '20) SHARP REGULARITY OF EXTERIOR COLLAPS. REGION

IF (K,E) GENERALIZED MINIMIZER OF $\Psi^{e}_{W}(v)$

$\mathcal{H}^n(\Sigma) = 0.$

THEN $\exists\ \Sigma \subseteq K \setminus E$ CLOSED SUCH THAT

1) $K \setminus \Sigma$ IS A SMOOTH & STABLE MINIMAL HYPERSF.

THM (KING M. STUVARD '20) SHARP REGULARITY OF EXTERIOR COLLAPS. REGION

IF (K,E) GENERALIZED MINIMIZER OF $\Psi^{e}_{W}(v)$

ALLARD

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THEN $\exists\ \Sigma \subseteq K \setminus E$ CLOSED SUCH THAT

1) $K \setminus \Sigma$ IS A SMOOTH & STABLE MINIMAL HYPERSF.

2) Σ IS $\begin{cases} \text{EMPTY} & \text{IF } 1 \leq n \leq 6 \\ \text{LOC. FINITE IN } \Omega \setminus \overline{E} & \text{IF } n=7 \\ \text{LOC. } (n-7) \text{ RECT. IN } \Omega \setminus \overline{E} & \text{IF } n \geq 8 \end{cases}$

PROOF: STEP ONE K IS AN ALMGREN MINIMIZER IN $\Omega \setminus E$

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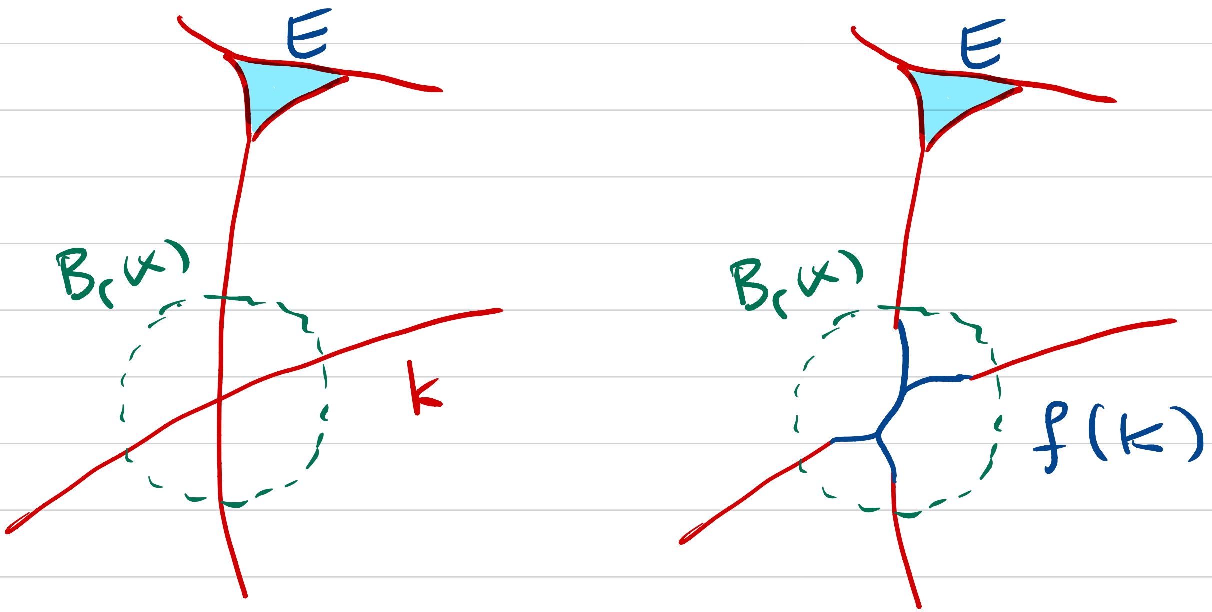
$$\mathcal{H}^n(K \cap B_r(x)) \leq \mathcal{H}^n(f(K) \cap B_r(x))$$

$\forall f: \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ LIPSCHITZ WITH $f(B_r(x)) \subseteq B_r(x) \subset\subset \Omega \setminus E$

$\{f \neq id\} \subset\subset B_r(x)$



RMK WE KNOW THIS WITH f DIFFEOMORPHISM

PROOF: **STEP ONE** K IS AN ALMGREN MINIMIZER IN $\Omega \setminus \overline{E}$

$\mathcal{H}^n(K \cap B_r(x)) \leq \mathcal{H}^n(f(K) \cap B_r(x))$ $\forall f : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ **LIPSCHITZ**

WITH $f(B_r(x)) \subseteq B_r(x) \subset\subset \Omega \setminus \overline{E}$

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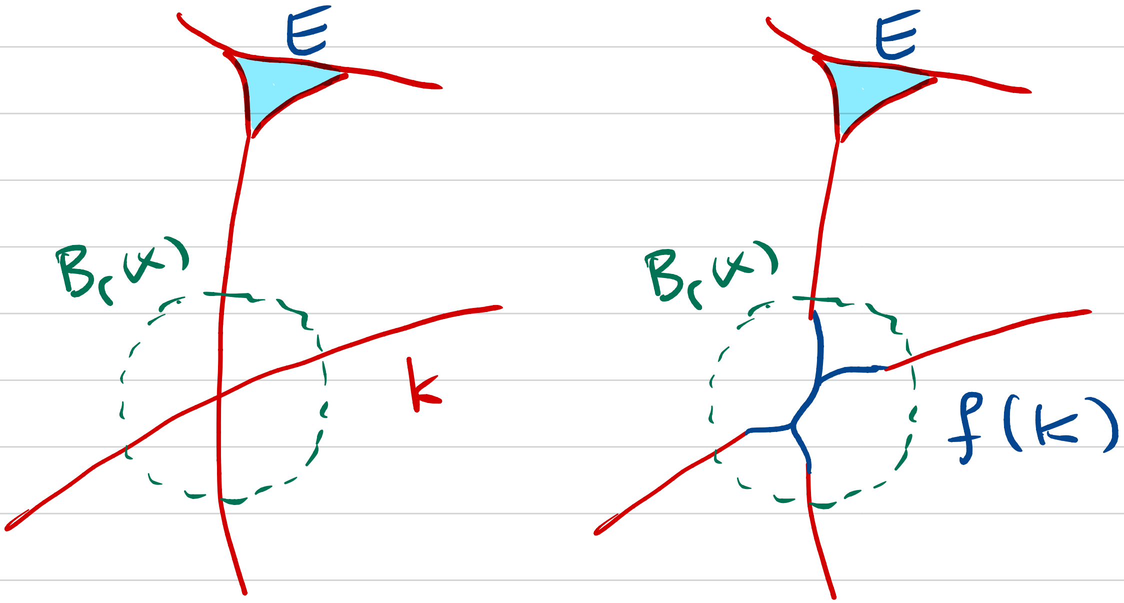

SINCE $\mathcal{E}(K, E) = \mathcal{H}^n(\Omega \cap \partial^* E) + 2\, \mathcal{H}^n(K \setminus \partial E)$

$\leq P(E_j, \Omega)$ $\forall E_j$ COMPETITOR OF $\psi^e_W(v)$

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$$\mathcal{H}^n(K \cap B_r(x)) \overset{\boxed{}}{\le} \mathcal{H}^n(f(K) \cap B_r(x))$$

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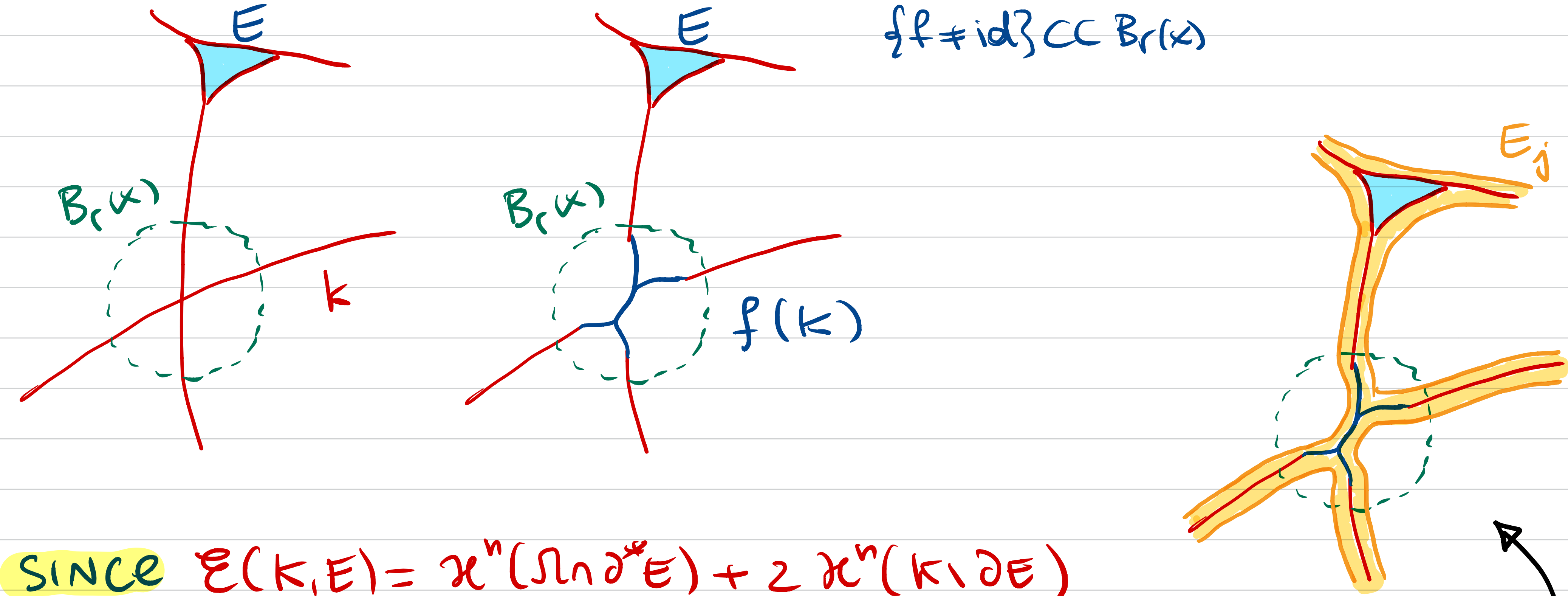

SINCE $\mathcal{E}(K,E) = \mathcal{H}^n(\Omega \cap \partial^* E) + 2\,\mathcal{H}^n(K \setminus \partial E)$

$\overset{\star}{\le} P(E_j, \Omega)$ $\forall\, E_j$ COMPETITOR OF $\Psi_W^e(v)$ WE NEED

& THEN $\star$ IMPLIES $\boxed{}$

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$$\mathcal{H}^n(K \cap B_r(x)) \overset{\square}{\le} \mathcal{H}^n(f(K) \cap B_r(x))$$

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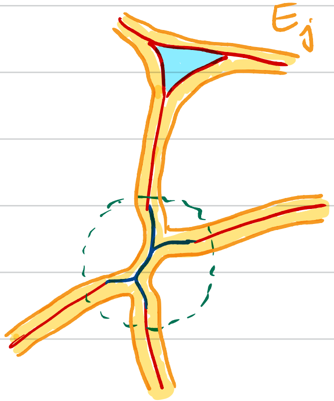
TO UNDERSTAND



WE NEED TO SHOW

$$\frac{\mathcal{L}^{n+1}(I_\eta(f(K) \cup E))}{2\eta} \approx P(E_j; \Omega) \longrightarrow 2\,\mathcal{H}^n(f(K) \setminus \partial^* E) + \mathcal{H}^n(\Omega \cap \partial^* E)$$

$\eta = \eta_j \downarrow 0^+$

PROOF: **STEP ONE** K IS AN ALMGREN MINIMIZER IN $\Omega \setminus E$

$$\mathcal{H}^n(K \cap B_r(x)) \overset{\square}{\leq} \mathcal{H}^n(f(K) \cap B_r(x))$$

$\forall\, f : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ LIPSCHITZ

WITH $f(B_r(x)) \subseteq B_r(x) \subset\subset \Omega \setminus E$

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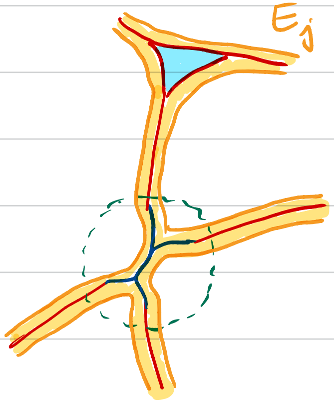
TO UNDERSTAND  WE NEED TO SHOW

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$\eta = \eta_j \downarrow 0^+$

DELICATE MINKOWSKI CONTENT CONSTRUCTION

- BASED ON EXTENSION OF AMBROSIO-COLESANTI-VILLA.

THM (KNESER '50's) IF $Z \subseteq \mathbb{R}^k$ & $f: \mathbb{R}^k \to \mathbb{R}^d$ LIPSCHITZ

THEN $f(Z)$ IS MINKOWSKI REGULAR, I.E.

$$\lim_{\eta \to 0^+} \frac{\mathcal{L}^d(I_\eta(f(Z)))}{\omega_{d-k}\,\eta^{d-k}} = \mathcal{H}^k(f(Z))$$

THM (AMBROSIO FUSCO PALLARA '00) IF $Z \subseteq \mathbb{R}^d$ COMPACT

& IF Z LOCALLY k-RECTIF. & $\mathcal{H}^k(Z \cap B_\rho(x)) \geq c\,\rho^k$, $\forall \rho < r_0$, $\forall x \in Z$

THEN Z IS MINKOWSKI REGULAR

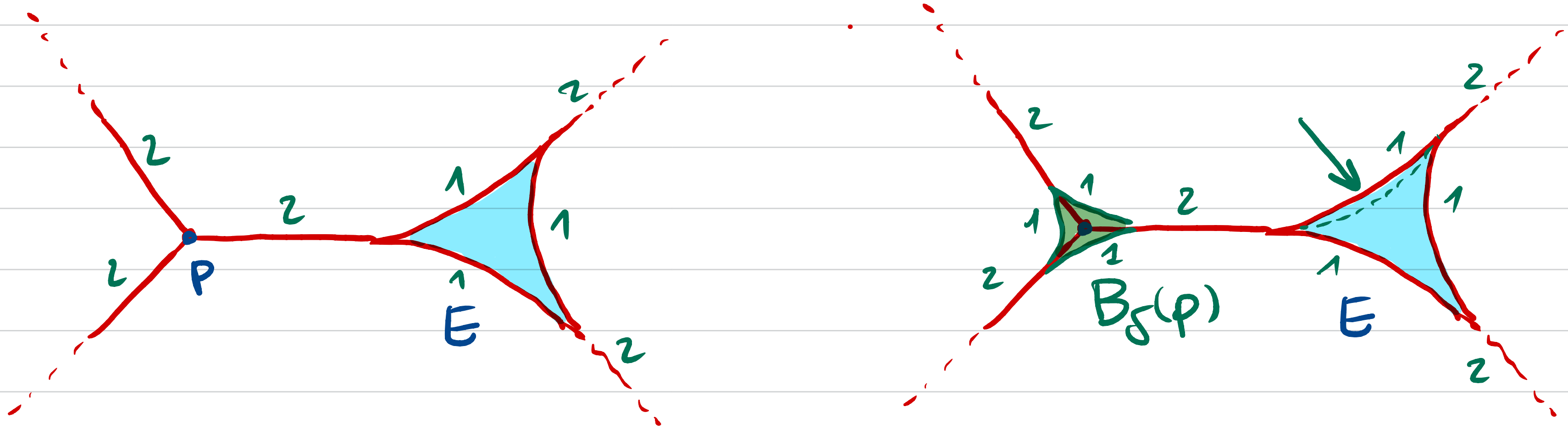
THM (KING M. STUVARD '20) IF $Z \subseteq \mathbb{R}^d$ COMPACT & k-RECTIFIABLE

IF $\begin{cases} \mathcal{H}^k(Z \cap B_\rho(x)) \geq c\,\rho^k, \\ \forall \rho < r_0, \forall x \in Z \end{cases}$ & IF $f: \mathbb{R}^d \to \mathbb{R}^d$ LIPSCHITZ

THEN $f(Z)$ IS MINKOWSKI REGULAR

STEP TWO $K \setminus E$ HAS NO Y-POINTS IN $\Omega \setminus E$.

PROOF BY "WETTING COMPETITORS" (NO LIP. IMAGES!)



1) WETTING $\to$ SAVE $O(\delta^n)$ IN AREA (MULT. 2 VS 1!)

2) REDUCE $O(\delta^{n+1})$ CHANGE IN VOLUME

BY PAYING $|\lambda| O(\delta^{n+1})$ IN PERIMETER

STEP THREE INVOKE REGULARITY THEORY

1) ALMGREN MINIMIZER $\oplus$ NO Y-POINTS

$\Rightarrow$ Σ OF $K \setminus \bar{E}$ IS $\mathcal{H}^{n-1}$-NEGLIGIBLE

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WITHOUT "CLASSICAL SINGULARITIES"

STEP THREE INVOKE REGULARITY THEORY

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WITHOUT "CLASSICAL SINGULARITIES"

$\Rightarrow$ BY WICKRAMASEKERA: THM PROVED IF $n \leq 7$.

& Σ IS $\mathcal{H}^{n-7+\eta}$-NEGLIGIBLE $\forall \eta > 0$ IF $n \geq 8$.

STEP THREE INVOKE REGULARITY THEORY

1) ALMGREN MINIMIZER $\oplus$ NO Y-POINTS

$\Rightarrow$ Σ OF $K \setminus \bar{E}$ IS $\mathcal{H}^{n-1}$-NEGLIGIBLE

2) $K \setminus \bar{E}$ DEFINES A MULT. ONE STATIONARY VARIFOLD WITHOUT "CLASSICAL SINGULARITIES"

$\Rightarrow$ BY WICKRAMASEKERA: THM PROVED IF $n \leq 7$.

& Σ IS $\mathcal{H}^{n-7+\eta}$-NEGLIGIBLE $\forall \eta > 0$ IF $n \geq 8$.

3) $n \geq 8$ NABER-VALTORTA.

THM (KING M. STUWARD '19)

IF (K,E) GENERALIZED MINIMIZER OF $\Psi^{e}_{W}(v)$

WITH EULER-LAGRANGE MULTIPLIER λ

I.E. $$\int_{\partial^* E} \operatorname{div}^K X + 2 \int_{K \setminus \partial^* E} \operatorname{div}^K X = \lambda \int_{\partial^* E} X \cdot \nu_E \qquad \forall X \in C^1_c(\Omega; \mathbb{R}^{n+1})$$

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THEN 1) IF $K\setminus \overline{E} \neq \emptyset$ THEN $\lambda<0$

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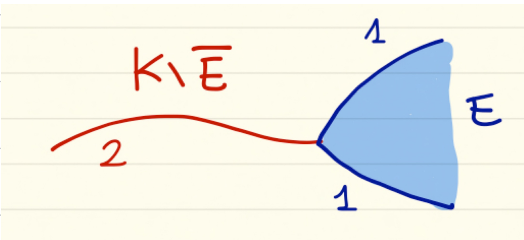
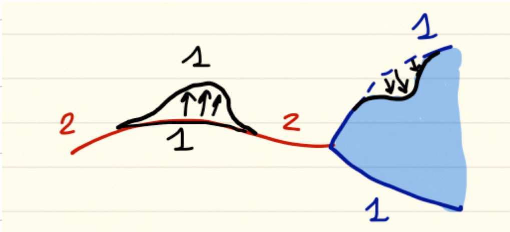
PROOF OF 1) REQUIRES

NON-MAPPING COMPETITORS





AREA INCREASE QUADRATIC IN ADDED VOLUME ε BY $H_{K \setminus E} = 0$.

FOLLOWED BY AREA VARIATION $-\lambda\varepsilon$ WHICH RESTORES VOLUME

OF COURSE $C\varepsilon^2 - \lambda\varepsilon > 0 \Rightarrow C > \frac{\lambda}{\varepsilon}$ AS $\varepsilon \to 0^+ \Rightarrow \lambda < 0$.