

A MESOSCALE FLATNESS CRITERION &
ITS APPLICATION TO EXTERIOR ISOPERIMETRY

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SUMMARY : WHILE STUDYING THE EXTERIOR ISOPERIMETRIC PROB.

WE HAVE FOUND NATURAL TO FORMULATE A

NEW FLATNESS CRITERION OPERATING AT MESOSCALES RATHER THAN

AT VERY SMALL OR VERY LARGE SCALES.

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WHAT IS A FLATNESS CRITERION?

ALLARD'S REGULARITY THM (EXAMPLE OF F.C.):

IF M HYPERSURFACE IN $\Omega \subseteq \mathbb{R}^{n+1}$ OPEN,

$|H_M| \leq \Lambda$ ON Ω & , GIVEN $x \in M$,

$\exists s > 0$ S.T. $s < \epsilon_0 / \Lambda$

$$\frac{\kappa^n(M \cap B_s(x))}{\omega_n s^n} \leq 1 + \epsilon_0$$

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THEN $\exists H$ HYPERTANE

$\exists u: H \rightarrow \mathbb{R}$ S.T.

$$M \cap B_{\epsilon s}(x) = \{y + u(y)v_H : y \in H\} \cap B_{\epsilon s}(x)$$

$$\|u\|_{C^1(H)} < \sigma$$

ALLARD'S REGULARITY THM (EXAMPLE OF F.C.): $\exists \epsilon_0 = \epsilon_0(n, \Lambda, \sigma)$ S.T.

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RMK: OBVIOUS THAT $\forall \Omega' \subset \subset \Omega \exists s_* = s_*(M, \Omega', \sigma)$ S.T., THE

ABOVE CONCLUSIONS HOLD WITH s_* IN PLACE OF $\epsilon_0 s$ & $H = T_x M, \forall x \in \Omega'$

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THE POINT OF A.R.T. IS RELATING FLATNESS SCALE TO A CONTROL QUANTITY WHICH IS STABLE BY AREA CONVERGENCE.

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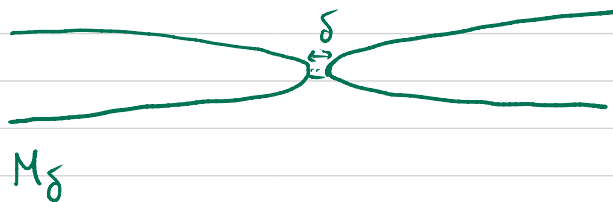
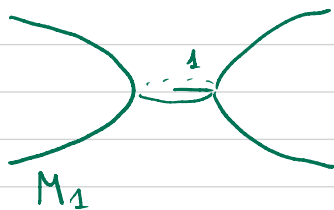
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RMK: ONE MUST RESORT TO A QUANTITY LIKE $\frac{\mathcal{H}^n(M \cap B_s(x))}{\omega_n s^n}$! CANNOT JUST USE Λ !

EX: CATENOIDS WITH WAIST SIZE δ [$H_M = 0 \ \forall \delta$]



IMPROVED CONVERGENCE A MODEL STATEMENT:

IF $\{M_j, \gamma_j, M\}$ (SMOOTH) HYPERSURFACES IN $\Omega \in \mathbb{R}^{n+1}$ OPEN

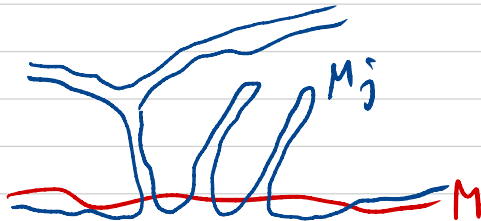
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THEN " $M_j \rightarrow M$ IN $C^1_{loc}(\Omega)$ "

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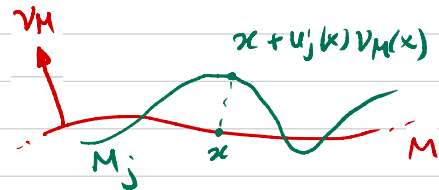
HP2 $\mathcal{H}^n(M_j \cap B_p(x)) \rightarrow \mathcal{H}^n(M \cap B_p(x)) \quad \forall x \in \Omega \text{ \& a.e. } p < \text{dist}(x, \partial\Omega)$

THEN " $M_j \rightarrow M$ IN $C^1_{loc}(\Omega)$ "

THAT IS $\forall \Omega' \subset\subset \Omega, j \geq j_{\Omega'}$, WE HAVE

$$M_j \cap \Omega' = \{x + u_j(x) \nu_M(x) : x \in M\} \cap \Omega',$$

$$\|u_j\|_{C^1(\Omega')} \rightarrow 0 \quad j \rightarrow \infty.$$



PROOF OF: IF $\{M_j, \nu_j, M$ (SMOOTH) HYPERSURFACES IN $\Omega \subseteq \mathbb{R}^{n+1}$ OPEN

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INDEED

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THEN " $M_j \rightarrow M$ IN $C^1_{\text{loc}}(\Omega)$ "

INDEED M SMOOTH $\Rightarrow \exists s_0 > 0$ S.T. $\mathcal{H}^n(M \cap B_{s_0}(x)) \leq (1 + \varepsilon/2) \omega_n s_0^n, \quad \forall x \in M \cap \Omega'$

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BY HP1 WE CAN APPLY ALLARD TO EACH M_j & TO M

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BY HP1 WE CAN APPLY ALLARD TO EACH M_j & TO $M \Rightarrow \exists H_x$ HYPERPLANE S.T.

M_j, M GRAPHS OF $f_j, f: H_x \rightarrow \mathbb{R}$ ON $B_{s_0}(x)$ & $f_j \rightarrow f$ IN C^1 BY HP2

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M_j, M GRAPHS OF $f_j, f: H_x \rightarrow \mathbb{R}$ ON $B_{\epsilon s_0}(x)$ & $f_j \rightarrow f$ IN C^1 BY HP2

CONCLUDE BY COVERING & REPARAMETRIZATION OF M_j OVER M

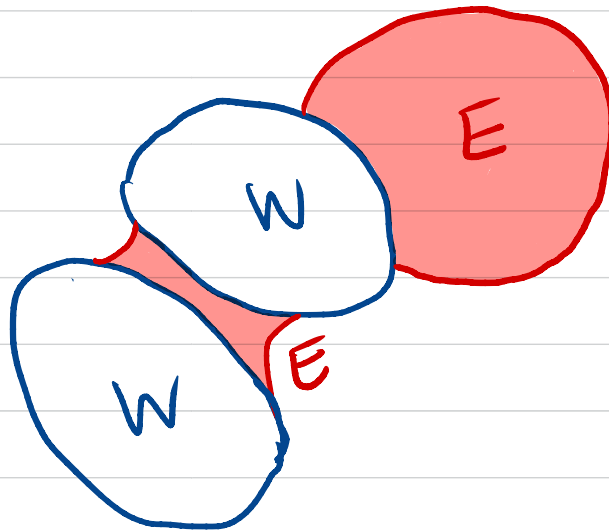
THE EXTERIOR ISOPERIMETRIC PROBLEM

DATA: W COMPACT SET $\subseteq \mathbb{R}^{n+1}$ $\sigma > 0$ VOLUME CONSTRAINT

PROBLEM: $\gamma_W(\sigma) = \inf \{ P(E; \Omega) : E \subseteq \Omega = \mathbb{R}^{n+1} \setminus W ; |E| = \sigma \}$

$P(E; \Omega) = \mathcal{H}^n(\Omega \cap \partial E) =$ PERIMETER OF E RELATIVE TO Ω

$|E| =$ VOLUME OF $E \subseteq \mathbb{R}^{n+1}$



THE EXTERIOR ISOPERIMETRIC PROBLEM

DATA: W COMPACT SET $\subseteq \mathbb{R}^{n+1}$ $\sigma > 0$ VOLUME CONSTRAINT

PROBLEM: $\psi_W(\sigma) = \inf \{ P(E; \Omega) : E \subseteq \Omega = \mathbb{R}^{n+1} \setminus W ; |E| = \sigma \}$

$$P(E; \Omega) = \mathcal{H}^n(\Omega \cap \partial E) \quad |E| = \text{VOLUME OF } E \subseteq \mathbb{R}^{n+1}$$

BASICS: MINIMIZERS EXIST $\forall \sigma > 0$

$\Omega \cap \partial E_\sigma$ HAS CONSTANT MEAN CURV. $H_{E_\sigma} = \psi'_W(\sigma)$ (FOR A.E. σ)

$\nu_{E_\sigma} \cdot \nu_W = 0$ IF $\partial W \in C^{1,1}$ (YOUNG'S LEMMA)

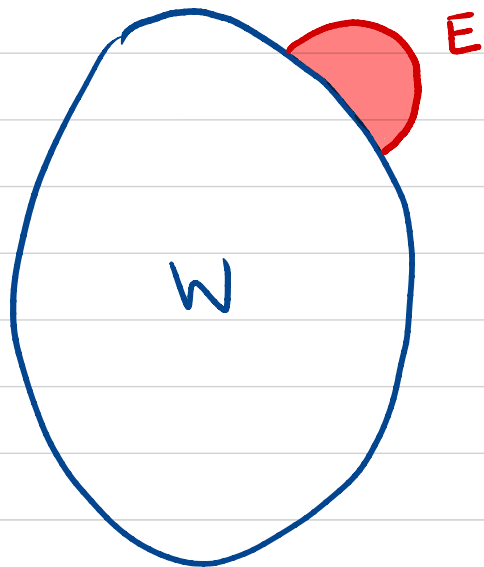
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ν SMALL



BASICS: MINIMIZERS EXIST $\forall \nu > 0$

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EXAMPLE $E_\nu \approx$ HALF BALL IF ν SMALL

$$H_{E_\nu} = O(\nu^{-1/(n+1)}) \quad \text{IF } \nu \rightarrow 0^+$$

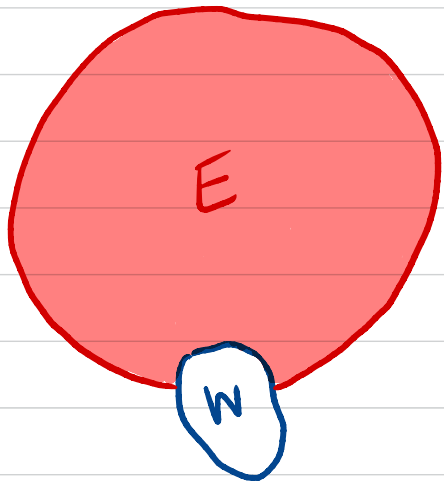
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PROBLEM: $\psi_W(v) = \inf \{ P(E; \Omega) : E \subseteq \Omega = \mathbb{R}^{n+1} \setminus W ; |E| = v \}$

$$P(E; \Omega) = \mathcal{H}^n(\Omega \cap \partial E) \quad |E| = \text{VOLUME OF } E \subseteq \mathbb{R}^{n+1}$$

v LARGE



BASICS: MINIMIZERS EXIST $\forall v > 0$ (FOR A.E. v)

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$\nu_{E_v} \cdot \nu_W = 0$ IF $\partial W \in C^{1,2}$ (YOUNG'S LAW)

EXAMPLE

$E_v \approx$ BALL IF v LARGE

$H_{E_v} = O(v^{-1/(n+1)})$ IF $v \rightarrow +\infty$

A CLOSER LOOK TO THE $\nu \rightarrow +\infty$ REGIME IF E_ν MINIMIZER OF $\Psi_\nu(\nu)$

$$E_\nu \approx B^{(\nu)} \text{ (BALL OF VOLUME } \nu) \quad H_{E_\nu} = O(\nu^{-\frac{1}{n+1}}) \text{ AS } \nu \rightarrow +\infty$$

A CLOSER LOOK TO THE $n \rightarrow \infty$ REGIME IF E_n MINIMIZER OF $\Psi_n(v)$

$E_n \approx B^{(n)}$ (BALL OF VOLUME n) $H_{E_n} = O(n^{-\frac{1}{n+1}})$ AS $n \rightarrow \infty$

COMPETITORS + ISOPERIMETRY : $\frac{\Psi_n(v)}{P(B^{(n)})} = \frac{\Psi_n(v)}{P(B^{(n)}) n^{-\frac{1}{n+1}}} \xrightarrow{(n \rightarrow \infty)} 1$

A CLOSER LOOK TO THE $\nu \rightarrow +\infty$ REGIME IF E_ν MINIMIZER OF $\Psi_W(\nu)$

$E_\nu \approx B^{(\nu)}$ (BALL OF VOLUME ν) $H_{E_\nu} = O(\nu^{-\frac{1}{n+1}})$ AS $\nu \rightarrow +\infty$

COMPETITORS + ISOPERIMETRY : $\frac{\Psi_W(\nu)}{P(B^{(\nu)})} = \frac{\Psi_W(\nu)}{P(B^{(\nu)}) \nu^{-\frac{1}{n+1}}} \xrightarrow{(\nu \rightarrow +\infty)} 1$

$$|E_\nu| = \nu \Rightarrow \underbrace{P(B^{(\nu)})}_{= O(\nu^{\frac{n}{n+1}})} \leq P(E_\nu) = \Psi_W(\nu) + \underbrace{P(E_\nu, \partial\Omega)}_{\leq \mathcal{H}^n(\partial\Omega)}$$

A CLOSER LOOK TO THE $\nu \rightarrow +\infty$ REGIME IF E_ν MINIMIZER OF $\Psi_W(\nu)$

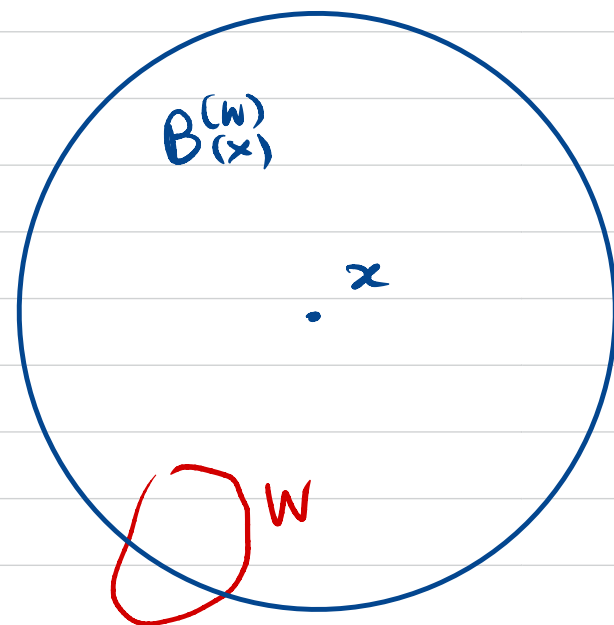
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IF W S.T. $|B_{(x)}^{(W)} \cap \Omega| = \nu$

THEN $\Psi_W(\nu) = P(E_\nu; \Omega) \leq P(B_{(x)}^{(W)}; \Omega)$
 $\approx P(B^{(\nu)}) - \kappa^n(W \cap \partial B_{(x)}^{(W)})$



A CLOSER LOOK TO THE $\nu \rightarrow +\infty$ REGIME IF E_ν MINIMIZER OF $\Psi_W(\nu)$

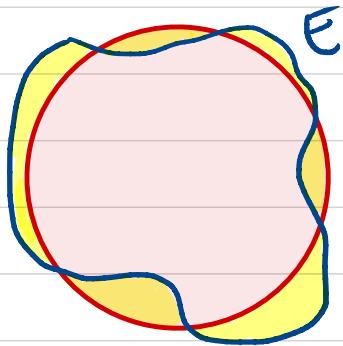
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QUANTITATIVE ISOPERIMETRY:

(FUSCO M. PRATELLI 06)

$$\frac{P(E)}{P(B^{(\nu)})} - 1 \geq c(n) \inf_{x \in \mathbb{R}^{n+1}} \left[\frac{|E \Delta B(x)|}{\nu} \right]^2$$



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QUANTITATIVE ISOPERIMETRY:

(FUSCO M. PRATELLI 06)

$$\frac{P(E)}{P(B^{(\nu)})} - 1 \geq c(n) \inf_{x \in \mathbb{R}^{n+1}} \left[\frac{|E \Delta B^{(\nu)}(x)|}{\nu} \right]^2 \Rightarrow \frac{|E_\nu \Delta B^{(\nu)}(x)|}{\nu} \leq \frac{C(n)}{\nu^{1/2(n+1)}}$$

A CLOSER LOOK TO THE $n \rightarrow \infty$ REGIME IF E_n MINIMIZER OF $\Psi_W(n)$

$$E_n \approx B^{(n)} \text{ (BALL OF VOLUME } n) \quad H_{E_n} = O(n^{-\frac{1}{n+1}}) \text{ AS } n \rightarrow \infty$$

COMPETITORS + ISOPERIMETRY:

$$\frac{\Psi_W(n)}{P(B^{(n)})} = \frac{\Psi_W(n)}{P(B^{(n)}) n^{-\frac{1}{n+1}}} \xrightarrow{(n \rightarrow \infty)} 1$$

QUANTITATIVE ISOPERIMETRY:

$$\frac{|E_n \Delta B^{(n)}(x)|}{n} \leq \frac{C(n)}{n^{\frac{1}{2(n+1)}}} \quad (*)$$

QUANTITATIVE ISOPERIMETRY

PLUS

IMPROVED CONVERGENCE / FLATNESS CRIT.

$$\exists R_0(n) \text{ [DEPEND. ON } n \text{ \& } W] \quad R_0(n) \rightarrow 0, \quad R_0(n) n^{\frac{1}{n+1}} \rightarrow \infty \quad (n \rightarrow \infty)$$

$$\partial E_n \setminus B_{R_0(n) n^{\frac{1}{n+1}}} \subseteq C^1\text{-SMALL GRAPH OVER } \partial B^{(n)}(x) \quad [x \text{ AS IN } (*)]$$

A CLOSER LOOK TO THE $n \rightarrow \infty$ REGIME IF E_n MINIMIZER OF $\Psi_W(v)$

$E_n \approx B^{(v)}$ (BALL OF VOLUME v) $H_{E_n} = O(v^{-\frac{1}{n+1}})$ AS $v \rightarrow \infty$

COMPETITORS + ISOPERIMETRY: $\frac{\Psi_W(v)}{P(B^{(v)})} = \frac{\Psi_W(v)}{P(B^{(v)}) v^{-\frac{1}{n+1}}} \xrightarrow{(v \rightarrow \infty)} 1$

QUANTITATIVE ISOPERIMETRY: $\frac{|E_n \Delta B^{(v)}(x)|}{v} \leq \frac{C(n)}{v^{\frac{1}{2(n+1)}}} (*)$

QUANTITATIVE ISOPERIMETRY

PLUS

IMPROVED CONVERGENCE / FLATNESS CRIT.

$\exists R_0(v)$ [DEPEND. ON n & W] $R_0(v) \rightarrow 0$, $R_0(v) v^{\frac{1}{n+1}} \rightarrow \infty$ ($v \rightarrow \infty$)

$\partial E_n \setminus B_{R_0(v) v^{\frac{1}{n+1}}} \subseteq C^1$ -SMALL GRAPH OVER $\partial B^{(v)}(x)$ [x AS IN $(*)$]

ALL NICE, BUT NO INFORMATION SPECIFIC TO THE OBSTACLE!

$$\psi_w(v) = P(\mathcal{E}_v, \Omega)$$

$$\approx P(B^{(v)})$$

AS $n \rightarrow \infty$

$$\partial \mathcal{E}_v \setminus B_{R_0(v)} v^{1/n+1}$$

IS C^1 -SMALL

NORMAL GRAPH

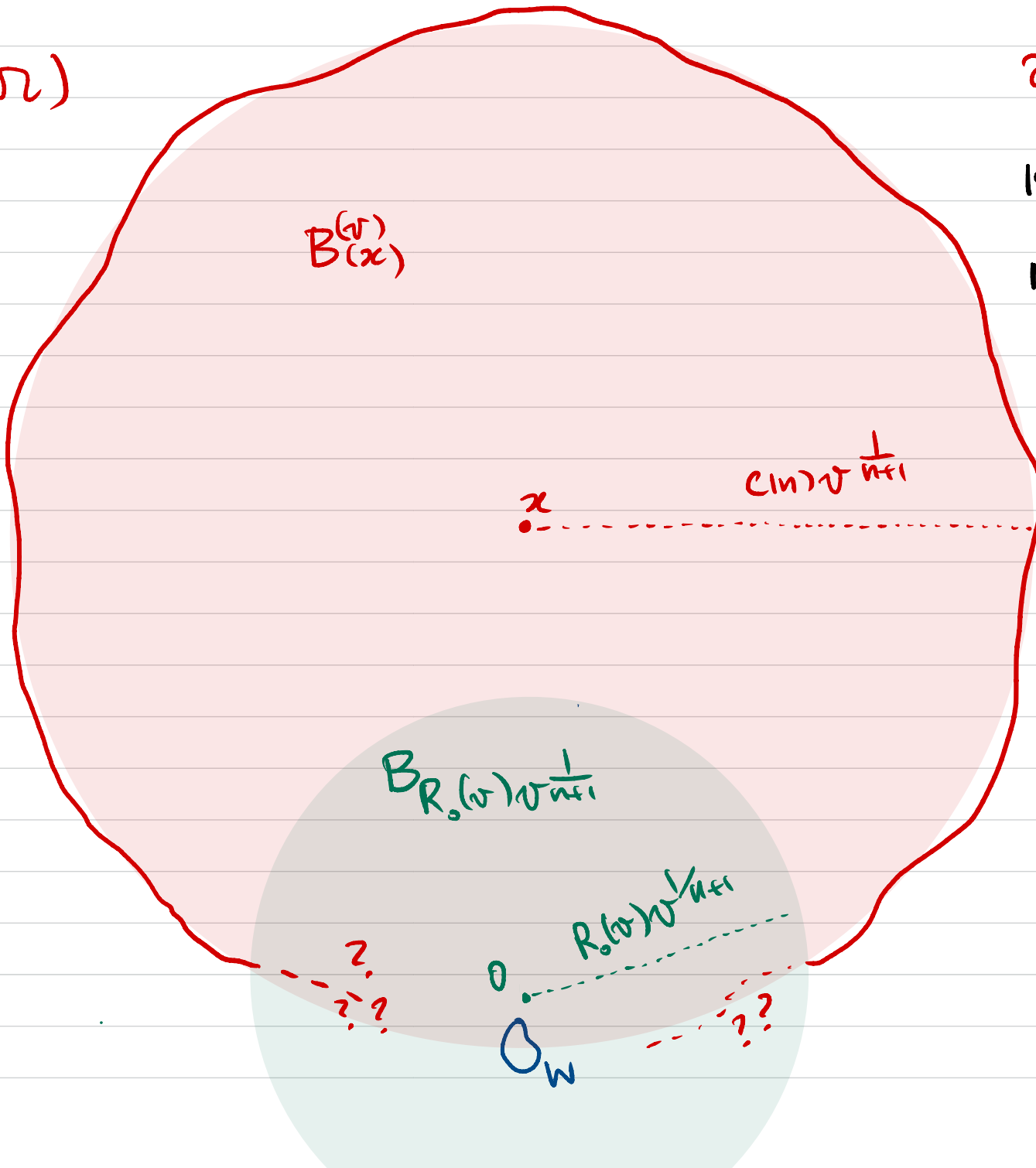
OVER $\partial B^{(v)}$

$$R_0(v) \rightarrow 0$$

$$R_0(v) v^{1/n+1} \rightarrow \infty$$

QUESTION:

CAN ONE SAY
SOMETHING
GENERAL &
RELATED
TO w ?



ISOPERIMETRIC RESIDUES

IDEA: LOOK AT L_{loc}^1 - SUBSEQUENTIAL LIMITS OF $\left\{ E_{\nu_j} \text{ MIN OF } \varphi_W(\nu_j) \right\}_{\nu_j \rightarrow \infty}$

ISOPERIMETRIC RESIDUES

IDEA: LOOK AT L_{loc}^1 - SUBSEQUENTIAL LIMITS OF $\left\{ E_{v_j} \text{ MIN OF } \varphi_W(v_j) \right\}_{v_j \rightarrow \infty}$

THEY EXIST BY STANDARD COMPACTNESS FOR SOFP

ISOPERIMETRIC RESIDUES

IDEA: LOOK AT L^1_{loc} -SUBSEQUENTIAL LIMITS OF $\left\{ E_{v_j} \text{ MIN OF } \varphi_W(w_j) \right\}_{v_j \rightarrow \infty}$

THEY EXIST BY STANDARD COMPACTNESS FOR SOFP

SO LET $E_{v_j} \rightarrow F$ IN $L^1_{loc}(\mathbb{R}^{n+1})$ (I.E. $| (F \Delta E_{v_j}) \cap B_R | \xrightarrow{j \rightarrow \infty} 0, \forall R > 0$)

ISOPERIMETRIC RESIDUES

IDEA: LOOK AT L^1_{loc} -SUBSEQUENTIAL LIMITS OF $\{E_{v_j} \text{ MIN OF } \varphi_W(w_j)\}_{v_j \rightarrow \infty}$

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BY $H_{E_{v_j}} = O(v_j^{-1/(n+1)})$ AS $v_j \rightarrow \infty$

$\Rightarrow H_F = 0$ MINIMAL BOUNDARY (∂F IS MINIMAL SURFACE)

ISOPERIMETRIC RESIDUES

IDEA: LOOK AT L^1_{loc} -SUBSEQUENTIAL LIMITS OF $\left\{ E_{v_j} \text{ MIN OF } \varphi_W(w_j) \right\}_{v_j \rightarrow \infty}$

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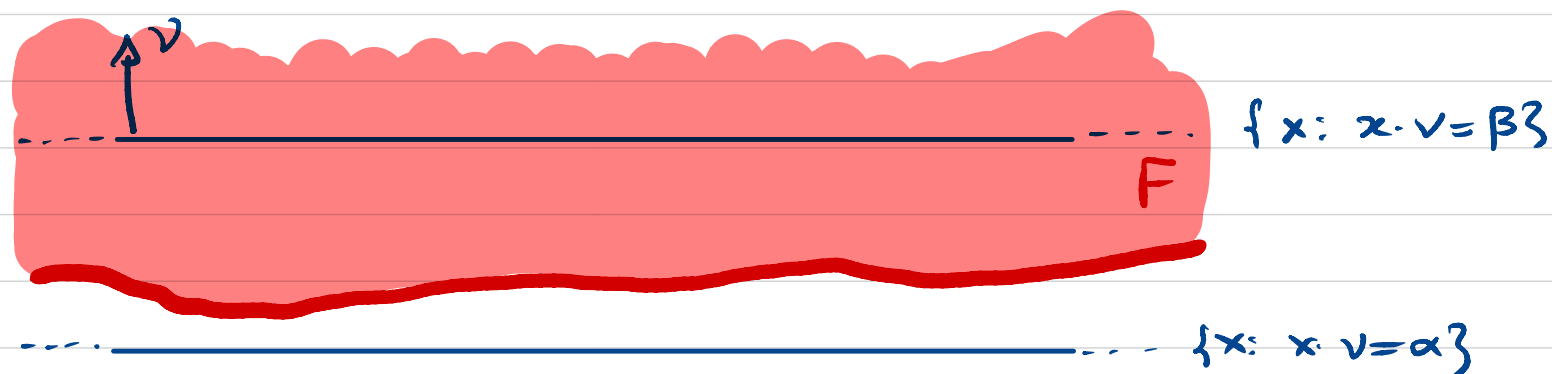
NOT SO USEFUL NEED A VARIATIONAL CHARACTERIZATION

ISOPERIMETRIC RESIDUES

CONSIDER $\mathcal{F} = \text{PAIRS } (F, \nu)$ S.T. $F \subseteq \mathbb{R}^{n+1}$, $\nu \in \mathcal{S}^n$ &

1) $\text{proj}_{\nu^\perp}(\partial F) = \nu^\perp$

2) $\partial F \subseteq \{x: \alpha < x \cdot \nu < \beta\}$ FOR SOME $\alpha, \beta \in \mathbb{R}$.



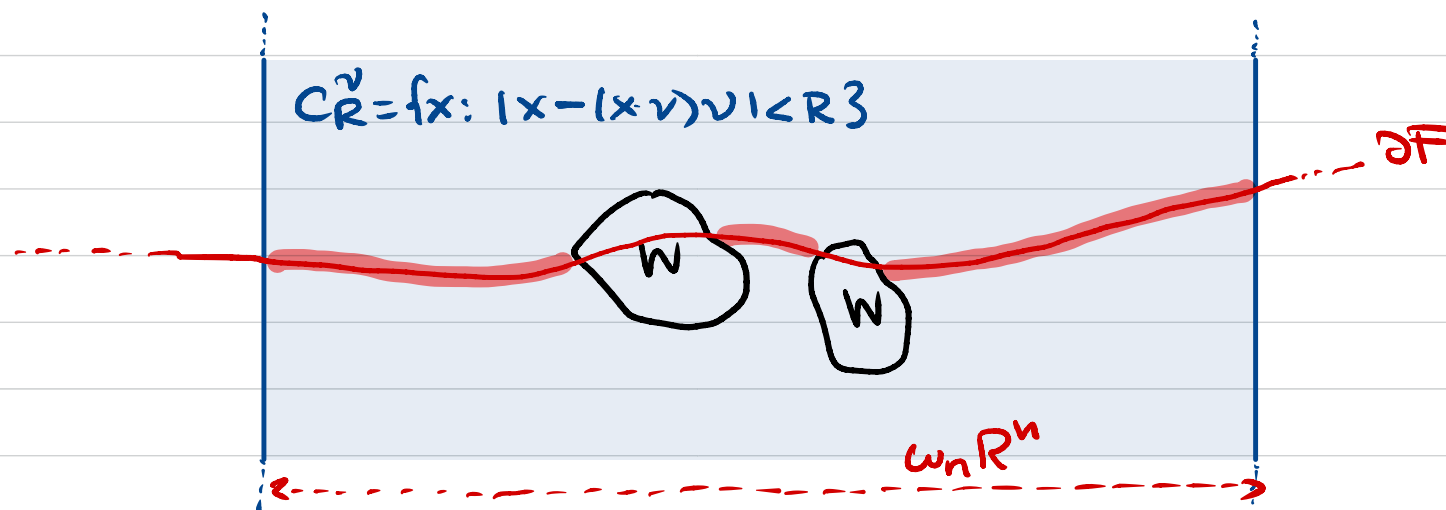
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1) $\text{proj}_{\nu^\perp}(\partial F) = \nu^\perp$ 2) $\partial F \subseteq \{x: \alpha < x \cdot \nu < \beta\} \quad \alpha, \beta \in \mathbb{R}.$

DEFINE W -RESIDUAL PERIMETER OF (F, ν)

$$\text{res}_W(F, \nu) = \lim_{R \rightarrow \infty} \omega_n R^n - P(F; C_R^\nu | W)$$



ISOPERIMETRIC RESIDUES

CONSIDER $\mathcal{F}$ = PAIRS (F, ν) S.T. $F \subseteq \mathbb{R}^{n+1}$, $\nu \in \mathcal{S}^n$ &

1) $\text{proj}_{\nu^\perp}(F) = \nu^\perp$ 2) $F \subseteq \{x: \alpha < x \cdot \nu < \beta\}$ $\alpha, \beta \in \mathbb{R}$.

DEFINE W -RESIDUAL PERIMETER OF (F, ν)

$$\text{res}_W(F, \nu) = \lim_{R \rightarrow \infty} \omega_n R^n - P(F; C_R^\nu | W)$$

DEFINE ISOPERIMETRIC RESIDUE OF W

$$R(W) = \sup \{ \text{res}_W(F, \nu) : (F, \nu) \in \mathcal{F} \}.$$

~ "PLATEAU'S PROB WITH FREE BOUNDARY ON W & AT ∞ "

$$(R(\lambda W) = \lambda^n R(W) \quad \forall \lambda > 0)$$

ISOPERIMETRIC RESIDUES

CONSIDER $\mathcal{F}$ = PAIRS (F, ν) S.T. $F \subseteq \mathbb{R}^{n+1}$, $\nu \in \mathcal{S}^n$ &

$$1) \text{proj}_{\nu^\perp}(F) = \nu^\perp \quad 2) F \subseteq \{x: \alpha < x \cdot \nu < \beta\} \quad \alpha, \beta \in \mathbb{R}.$$

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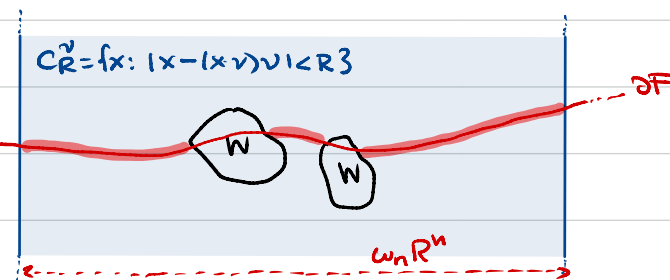
$$R(W) = \sup \{ \text{res}_W(F, \nu) : (F, \nu) \in \mathcal{F} \}.$$

~ "PLATEAU'S PROB WITH FREE BOUNDARY ON W & AT ∞ "

~ CLOSE TO A SECTIONAL AREA OF W (BUT NOT QUITE THAT!)

ISOPERIMETRIC RESIDUES

$$\pi_{\text{es}}_W(F, \nu) = \lim_{R \rightarrow \infty} \omega_n R^n - P(F; C_R^\nu | W)$$



ISOPERIMETRIC RESIDUE OF W

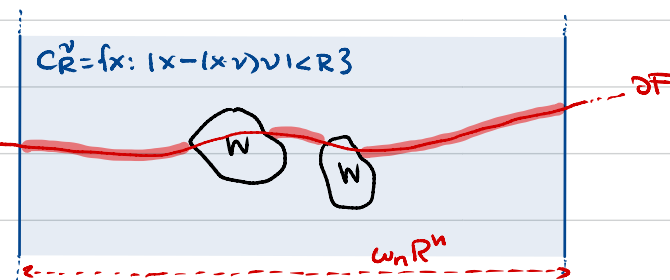
$$R(W) = \sup \{ \pi_{\text{es}}_W(F, \nu) : (F, \nu) \in \mathcal{F} \}.$$

Then

$$1) \quad \psi_W(n) = P(B^{(n)}) - R(W) + o(1) \quad \text{AS } n \rightarrow +\infty$$

ISOPERIMETRIC RESIDUES

$$\text{res}_W(F, \nu) = \lim_{R \rightarrow \infty} \omega_n R^n - P(F; C_R^\nu | W)$$



ISOPERIMETRIC RESIDUE OF W

$$R(W) = \sup \{ \text{res}_W(F, \nu) : (F, \nu) \in \mathcal{F} \}.$$

Then 1) $\psi_W(n) = P(B^{(n)}) - R(W) + o(1)$ AS $n \rightarrow +\infty$

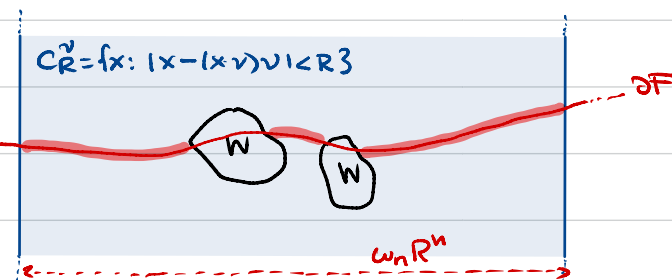
2) $\exists R_2 = R_2(n, W)$ S.T. $\forall E_n$ MINIMIZER OF $\psi_W(n)$ $\forall R > 0$

$\exists$ MAXIMIZER F OF $R(W)$ SUCH THAT

$\partial E_n \cap (B_R \setminus B_{R_2}) \in C^1$ -SMALL GRAPH OVER ∂F ($n > n(n, W, R)$)

ISOPERIMETRIC RESIDUES

$$\text{res}_W(F, \nu) = \lim_{R \rightarrow \infty} \omega_n R^n - P(F; C_R^\nu | W)$$



ISOPERIMETRIC RESIDUE OF W

$$R(W) = \sup \{ \text{res}_W(F, \nu) : (F, \nu) \in \mathcal{F} \}.$$

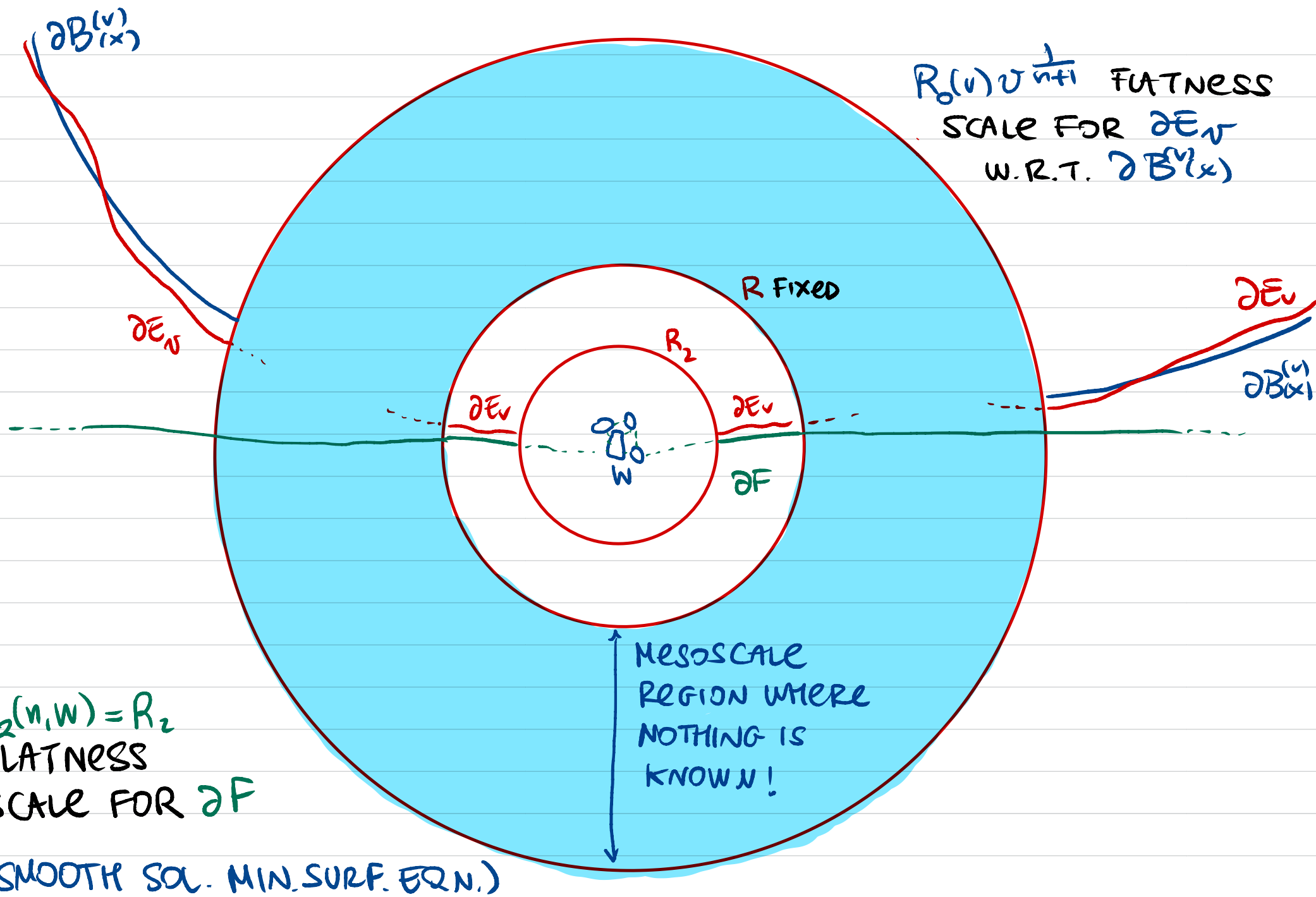
Then 1) $\psi_W(n) = P(B^{(n)}) - R(W) + o(1)$ AS $n \rightarrow +\infty$

2) $\exists R_2 = R_2(n, W) : \forall E \text{ MIN } \psi_W(n) \forall R > 0 \exists \text{ MAXIMIZER } F \text{ OF } R(W)$

$\partial E_n \cap (B_R, B_{R_2}) \in C^1$ - SMALL GRAPH OVER ∂F ($n > n_0(n, W, R)$)

RMK: PROOFS OF 1) & 2) REQUIRE BLOW-DOWN & DECAY ANALYSIS

FOR EXTERIOR MINIMAL SURFACES (EXTERIOR FLATNESS CRITERION)



BY OUR NEW

MESOSCALE FLATNESS CRITERION

$\Rightarrow$

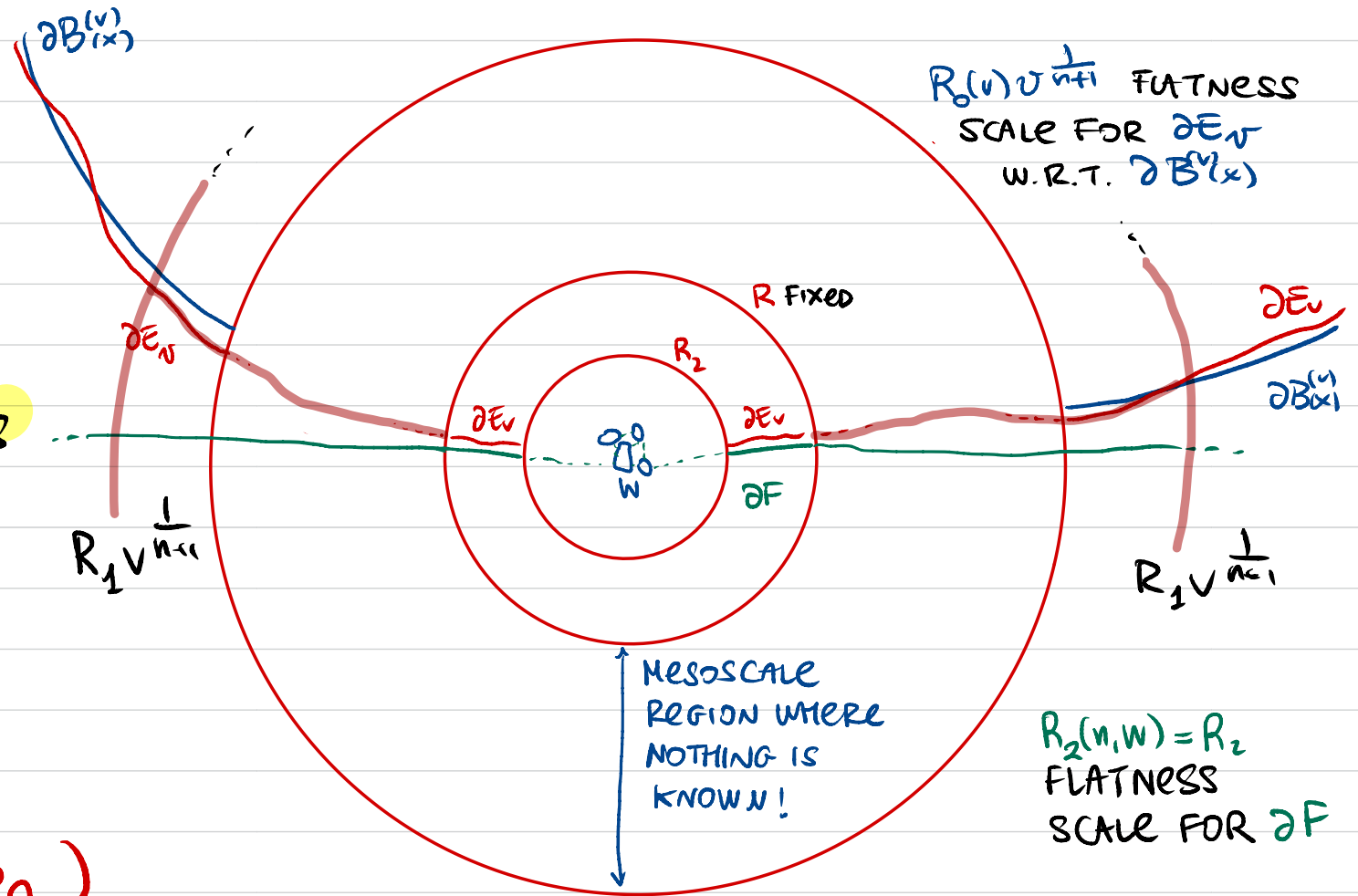
$\exists R_1 = R_1(n, W)$ S.T.

$$\partial E_N \cap (B_{R_1 \sqrt{\frac{1}{n+1}}} \setminus B_{R_2})$$

$\subseteq C^2$ -SMALL GRAPH OVER ∂F

IN PARTICULAR FOR N LARGE

$$R_1 \sqrt{\frac{1}{n+1}} > R_0(n) \sqrt{\frac{1}{n+1}} \Rightarrow \text{OVERLAPPING!}$$

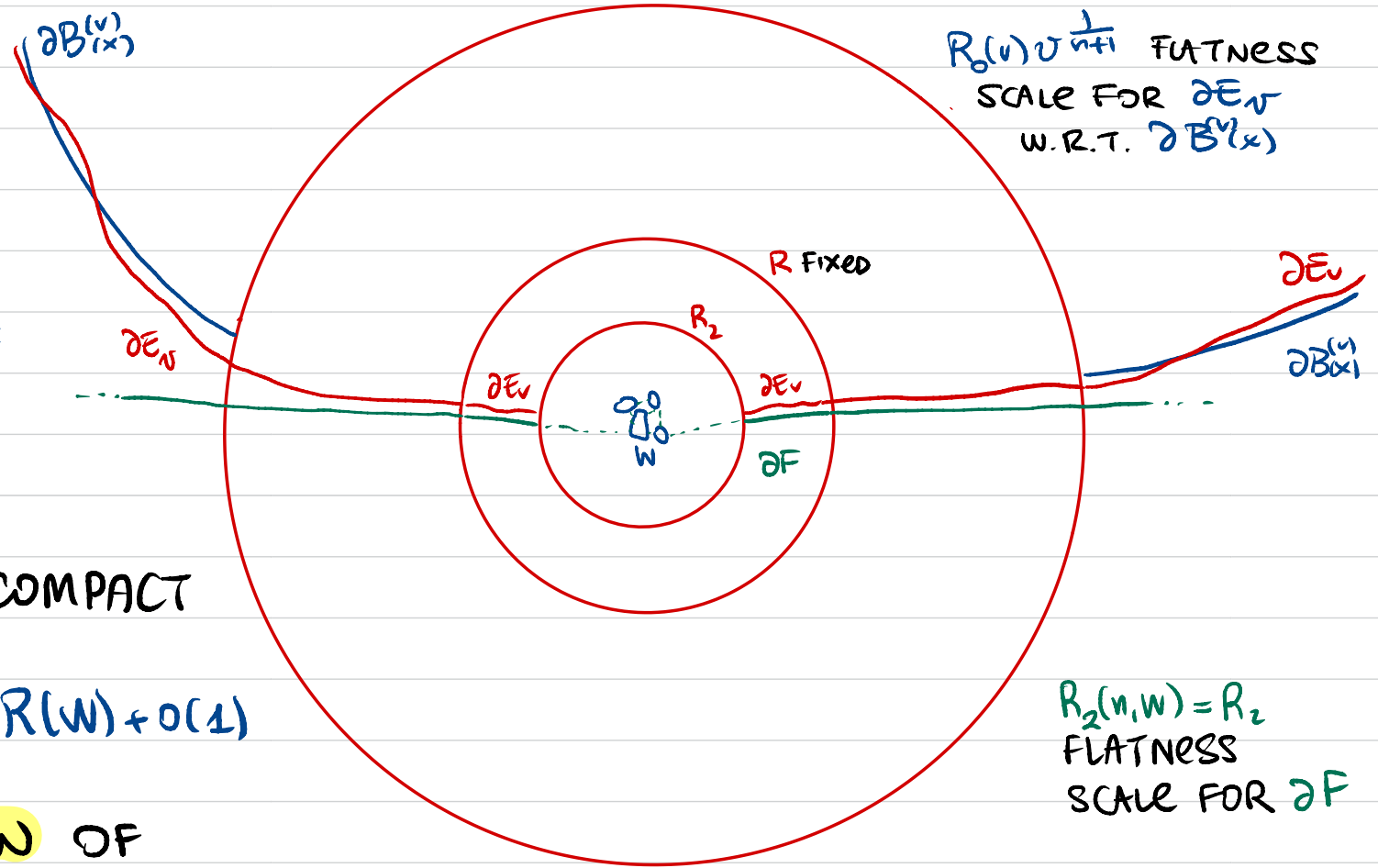


MESOSCALE FLATNESS
CRITERION $\Rightarrow$

$\exists R_1 = R_1(n, W)$ S.T.

$\partial E_N \cap (B_{R_1 V^{\frac{1}{n+1}}} \setminus B_{R_2})$

$\subseteq C^1$ -SMALL GRAPH OVER ∂F



THEOREM 1 IF W COMPACT

THEN $\psi_W(r) = P(B^N_r) - R(W) + o(1)$

& FULL RESOLUTION OF

EXTERIOR ISOPERIMETRIC SETS (UP TO W) USING THE MODELS

$\partial B^N(x)$ & ∂F

IN PARTICULAR, $\partial E_N \setminus B_{R_2}$ IS AN n -DIMENSIONAL DISK.

NOW:

1) PROPERTIES OF ISOPERIMETRIC RESIDUES

2) MESOSCALE FLATNESS CRITERION

THEOREM 2 $W \subseteq \mathbb{R}^{n+1}$ COMPACT

$$1) \quad \mathcal{I}(W) \leq \mathcal{R}(W) \leq \mathcal{P}(W) \quad \sup \{ \mathcal{H}^n(\text{proj}_{v^\perp}(W)) : v \in \mathbb{S}^n \}$$

$$\sup \{ \mathcal{H}^n(W \cap \Pi) : \Pi \text{ hyperplane} \}$$

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RMK: $\mathcal{R}(\lambda W) = \lambda^n \mathcal{R}(W) \quad \forall \lambda > 0$

THEOREM 2 $W \subseteq \mathbb{R}^{n+1}$ COMPACT

1) $\mathcal{J}(W) \leq \mathcal{R}(W) \leq \mathcal{P}(W)$

2) $\mathcal{R}(W) = 0 \Leftrightarrow W$ PURELY $\mathcal{H}^n$ -UNRECTIFIABLE ($\mathcal{H}^n(W) < \infty$)

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3) $\mathcal{R}(W) \leq \mathcal{R}(\text{Ball diam } W)$ + CHARACT. EQUALITY CASES

a) W CONTAINS AN $(n-1)$ -DIM SPHERE OF DIAM = $\text{diam}(W)$

b) $\Omega \setminus [\text{EXTERNAL HYP TO THAT SPHERE}]$ HAS 2 UNBOUNDED COMP.

THEOREM 2 IF $W \subseteq \mathbb{R}^{n+1}$ COMPACT, THEN

1) $\mathcal{I}(W) \leq \mathcal{R}(W) \leq \mathcal{P}(W)$

2) $\mathcal{R}(W) = 0 \iff W$ PURELY $\mathcal{H}^n$ -UNRECTIFIABLE ($\mathcal{H}^n(W) < \infty$)

3) $\mathcal{R}(W) \leq \mathcal{R}(\text{Ball diam } W) + \text{CHARACT. EQUALITY CASES}$

4) $\exists R_2 = R_2(n, W)$ S.T. IF (F, ν) MAXIMIZER OF $\mathcal{R}(W)$, THEN

$\partial F \setminus B_{R_2} = \text{GRAPH OVER } \nu^\perp \text{ OF } f \text{ SOL MIN SURF EQN}$

$$f(x) \equiv a \quad (n=1)$$

$$\left| f(x) - \left(a + \frac{b}{|x|^{n-2}} + \frac{c \cdot x}{|x|^n} \right) \right| \leq \frac{C_0}{|x|^n} \quad (n \geq 2)$$

$$|x|^{n-1} |\nabla f(x)|, |x|^n |\nabla^2 f(x)| \leq C_0$$

A MESOSCALE FLATNESS CRITERION

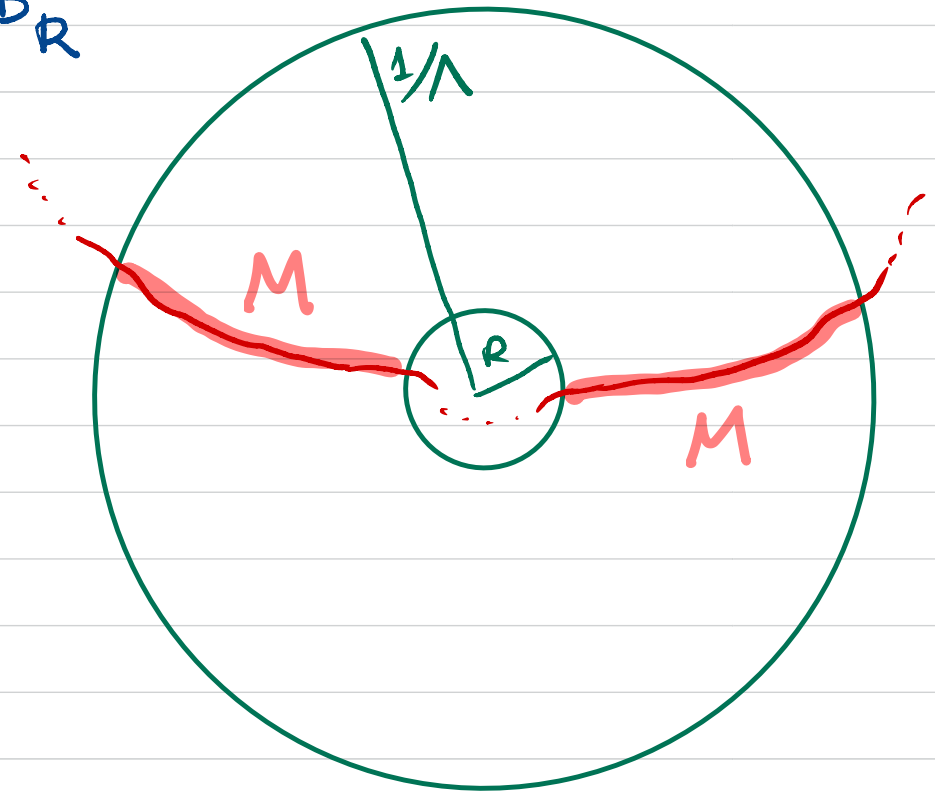
$\mathcal{M}_{\Lambda, R}$ = HYPERSURFACES M IN $\mathbb{R}^{n+1} \setminus B_R$

$$|H_M| \leq \Lambda \quad \text{ON } B_{1/\Lambda} \setminus B_R$$

$$\text{bdry}(M) \subseteq \partial B_R$$

TWO NOTIONS OF FLATNESS

- 1) ANGULAR FLATNESS
- 2) AREA DEFICIT



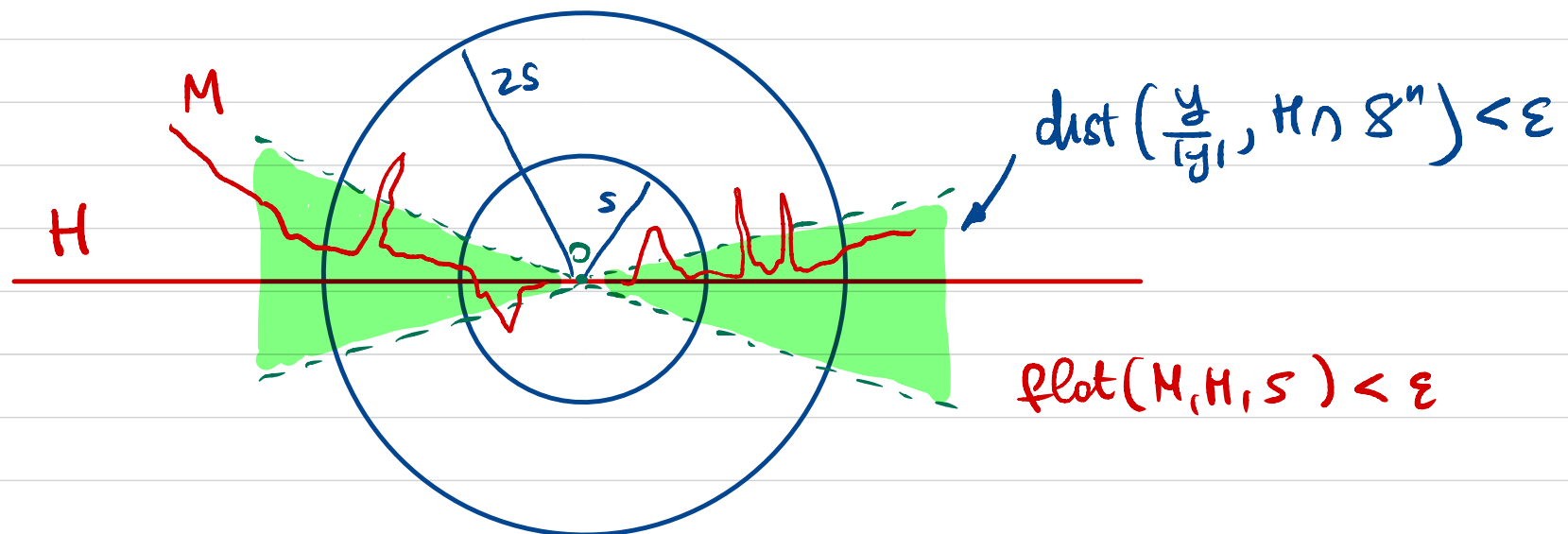
A MESOSCALE FLATNESS CRITERION

$M \in \mathcal{M}_{\Lambda, R}$ i.e. $|H_M| \leq \Lambda$ ON $B_{1/\Lambda} \setminus B_R$ & $\text{bdry}(M) \subseteq \partial B_R$

1) ANGULAR FLATNESS

$\mathcal{H}$ = HYPERPLANES THROUGH ORIGIN $H \in \mathcal{H}$

$$\text{flat}(M, H, s) = \frac{1}{s^n} \int_{M \cap (B_{2s} \setminus B_s)} \text{dist}_{\mathbb{R}^n}^2 \left(\frac{y}{|y|}, H \cap \mathbb{S}^{n-1} \right) d\mathcal{H}^n(y).$$



A MESOSCALE FLATNESS CRITERION

$M \in \mathcal{M}_{\Lambda, R}$ i.e. $|H_M| \leq \Lambda$ on $B_{1/\Lambda} \setminus B_R$ & $\text{bdry}(M) \subseteq \partial B_R$

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2) AREA DEFICIT: THE QUANTITY

$$\sigma_{M, \Lambda, R}(r) = \frac{\mathcal{H}^n(M \cap (B_r \setminus B_R))}{r^n}$$

A MESOSCALE FLATNESS CRITERION

$M \in \mathcal{M}_{\Lambda, R}$ i.e. $|H_M| \leq \Lambda$ on $B_{1/\Lambda} \setminus B_R$ & $\text{bdry}(M) \subseteq \partial B_R$

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2) AREA DEFICIT: THE QUANTITY

$$\sigma_{M, \Lambda, R}(r) = \frac{\mathcal{H}^n(M \cap (B_r \setminus B_R))}{r^n} + \Lambda \int_R^r \frac{\mathcal{H}^n(M \cap (B_\rho \setminus B_R))}{\rho^n} d\rho$$

A MESOSCALE FLATNESS CRITERION

$M \in \mathcal{M}_{\Lambda, R}$ i.e. $|H_M| \leq \Lambda$ on $B_{1/\Lambda} \setminus B_R$ & $\text{bdry}(M) \subseteq \partial B_R$

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2) AREA DEFICIT: THE QUANTITY

$$\Theta_{M, \Lambda, R}(\tau) = \frac{\mathcal{H}^n(M \cap (B_\tau \setminus B_R))}{\tau^n} - \frac{1}{n\tau^n} \int_{M \cap \partial B_R} x \cdot \nu_M^\infty d\mathcal{H}^{n-1} + \Lambda \int_R^\tau \frac{\mathcal{H}^n(M \cap (B_\rho \setminus B_R))}{\rho^n} d\rho$$

IS INCREASING IN $\tau \in (R, 1/\Lambda)$

A MESOSCALE FLATNESS CRITERION

$M \in \mathcal{M}_{\Lambda, R}$ i.e. $|H_M| \leq \Lambda$ ON $B_{1/\Lambda} \setminus B_R$ & $\text{bdry}(M) \subseteq \partial B_R$

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IS INCREASING IN $\pi \in (R, 1/\Lambda)$ &

IF $\Theta_{M, \Lambda, R} \equiv \text{CONSTANT}$ ON (π_1, π_2) THEN $M \cap (B_{\pi_2} \setminus B_{\pi_1}) = \text{A CONE.}$

A MESOSCALE FLATNESS CRITERION

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IS INCREASING IN $\pi \in (R, 1/\Lambda)$ &

IF $\Theta_{M, \Lambda, R} \equiv \text{CONSTANT}$ ON (π_1, π_2) THEN $M \cap (B_{\pi_2} \setminus B_{\pi_1}) = \text{A CONE}.$

RMK: $\Theta_{M, \Lambda, R} \equiv \omega_n$ FOR HYPERPLANES

$$\delta_{M, \Lambda, R}(\pi) = \omega_n - \Theta_{M, \Lambda, R}(\pi)$$

A MESOSCALE FLATNESS CRITERION

$M \in \mathcal{M}_{\Lambda, R}$ i.e. $|H_M| \leq \Lambda$ on $B_{1/\Lambda} \setminus B_R$ & $\text{bdry}(M) \subseteq \partial B_R$

1) ANGULAR FLATNESS $\text{flat}(M, H, s) = \frac{1}{s^n} \int_{M \cap (B_{2s} \setminus B_s)} \text{dist}_{\mathbb{R}^n}^2\left(\frac{y}{|y|}, H \cap \mathbb{S}^n\right) d\mathcal{H}^n(y).$

2) AREA DEFICIT: $\delta_{M, \Lambda, R}^{(2)} = \omega_n - \Theta_{M, \Lambda, R}(2)$

$$s < \frac{\varepsilon}{\Lambda}$$

A MESOSCALE FLATNESS CRITERION

$M \in \mathcal{M}_{\Lambda, R}$ i.e. $|H_M| \leq \Lambda$ ON $B_{1/\Lambda} \setminus B_R$ & $\text{bdry}(M) \subseteq \partial B_R$

1) ANGULAR FLATNESS $\text{flat}(M, H, s) = \frac{1}{s^n} \int_{M \cap (B_{2s} \setminus B_s)} \text{dist}_{\mathbb{R}^n}^2\left(\frac{y}{|y|}, H \cap \mathbb{S}^n\right) dH^n(y).$

2) AREA DEFICIT: $\delta_{M, \Lambda, R}^{(2)}(r) = \omega_n - \Theta_{M, \Lambda, R}(r)$

CRITERION $\forall \varepsilon \exists \varepsilon_0 : \text{IF } \exists s \gg R, s \ll 1/\Lambda \text{ (} s = \text{MESOSCALE) }$

SUCH THAT $\text{flat}(M, H, s) \leq \varepsilon_0$ & $|\delta_{M, \Lambda, R}(s)| \leq \varepsilon_0$

THEN $M \cap (B_{S_*} \setminus B_s)$ is σ -FLAT IN $\mathbb{C}^1$ W.R.T. H

$$S_* = \max\left\{\frac{\varepsilon_0}{\Lambda}, R_*\right\} \quad R_* = \sup\{p \geq s : \delta_{M, \Lambda, R}(p) \geq -\varepsilon_0\}.$$

A MESOSCALE FLATNESS CRITERION

$M \in \mathcal{M}_{\Lambda, R}$ i.e. $|H_M| \leq \Lambda$ ON $B_{1/\Lambda} \setminus B_R$ & $\text{bdry}(M) \subseteq \partial B_R$

1) ANGULAR FLATNESS $\text{flat}(M, H, s) = \frac{1}{s^n} \int_{M \cap (B_{2s} \setminus B_s)} \text{dist}_{\mathbb{R}^n}^2 \left(\frac{y}{|y|}, H \cap \mathbb{S}^n \right) dH^n(y).$

2) AREA DEFICIT: $\delta_{M, \Lambda, R}(r) = \omega_n - \Theta_{M, \Lambda, R}(r)$

CRITERION $\forall \epsilon \exists \epsilon_0 : \text{IF } \exists s \gg R, s \ll 1/\Lambda \quad (s = \text{MESOSCALE})$

SUCH THAT $\text{flat}(M, H, s) \leq \epsilon_0$ & $|\delta_{M, \Lambda, R}(s)| \leq \epsilon_0$

THEN $M \cap (B_{S_*} \setminus B_s)$ is σ -FLAT IN $\mathbb{C}^1$ W.R.T. H

$$S_* = \max \left\{ \frac{\epsilon_0}{\Lambda}, R_* \right\} \quad \underbrace{R_* = \sup \{ \rho \geq s : \delta_{M, \Lambda, R}(\rho) \geq -\epsilon_0 \}}_{\text{MAXIMAL GRAPHICAUTY PROPAGATION SCALE}}.$$

MAXIMAL GRAPHICAUTY PROPAGATION SCALE

A MESOSCALE FLATNESS CRITERION

$M \in \mathcal{M}_{\Lambda, R}$ i.e. $|H_M| \leq \Lambda$ ON $B_{1/\Lambda} \setminus B_R$ & $\text{bdry}(M) \subseteq \partial B_R$

CRITERION $\forall \epsilon \exists \epsilon_0$: IF $\exists s \gg R, s \ll 1/\Lambda$ ($s = \text{MESOSCALE}$)

SUCH THAT $\text{flat}(M, H, s) \leq \epsilon_0$ & $|\delta_{M, \Lambda, R}(s)| \leq \epsilon_0$

THEN $M \cap (B_{S_*} \setminus B_s)$ IS σ -FLAT IN C^1 W.R.T. H

$$S_* = \max\left\{\frac{\epsilon_0}{\Lambda}, R_*\right\} \quad R_* = \sup\{p \geq s : \delta_{M, \Lambda, R}(p) \geq -\epsilon_0\}.$$

SHARP FOR A GENERIC ELEMENT OF $\mathcal{M}_{\Lambda, R}$ FLATNESS CAN BE

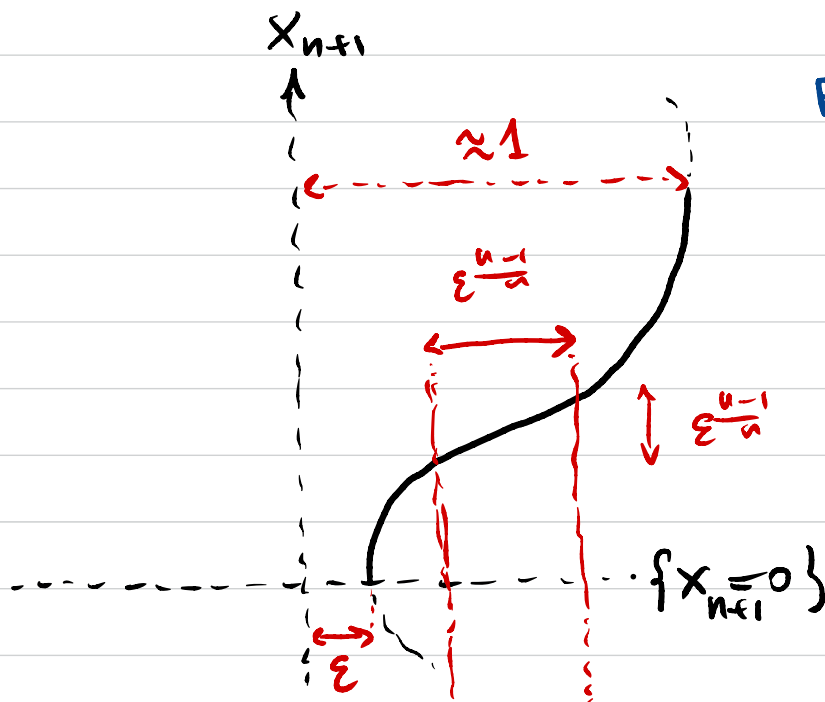
ESTABLISHED ONLY AT A MESOSCALE $s \gg R$ & $s \ll 1/\Lambda$

A MESOSCALE FLATNESS CRITERION

$M \in \mathcal{M}_{\Lambda, R}$ i.e. $|H_M| \leq \Lambda$ ON $B_{1/\Lambda} \setminus B_R$ & $\text{bdry}(M) \subseteq \partial B_R$

SHARP FOR A GENERIC ELEMENT OF $\mathcal{M}_{\Lambda, R}$ FLATNESS CAN BE ESTABLISHED ONLY AT A MESOSCALE $s \gg R$ & $s \ll 1/\Lambda$

EXAMPLE UNDULOIDS $\text{CMC} = n$ (LIKE UNITBALLS) & WAIST SIZE ε



FLATNESS W.R.T. $\{x_{n+1} = 0\}$

$$|\nabla f|(r) \approx O(\varepsilon^{(n-1)/n}) \quad \text{FOR } r = O(\varepsilon^{(n-1)/n})$$

$$1 \gg \varepsilon^{\frac{n-1}{n}} \gg \varepsilon \quad (n \geq 2)$$