AN INTRODUCTION TO BRANCHED TRANSPORT

(Or maybe an invitation to the theory of currents?)

FRG - NEW CHALLENGES IN GMT
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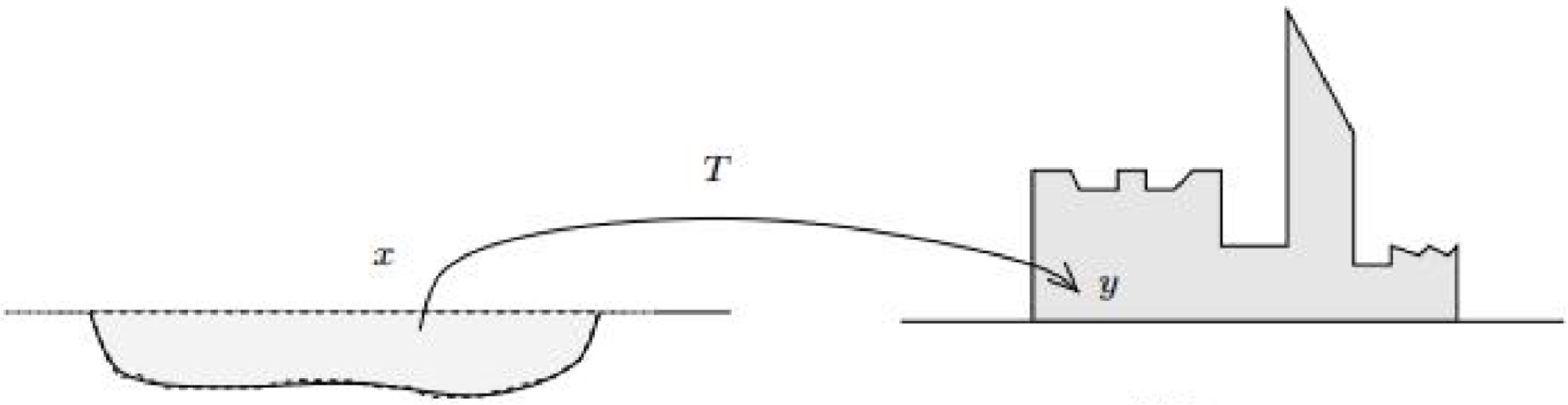
MONGE'S PROBLEM



RUBBLE: $(m_1, \dots, m_n)$ located at $(x_1, \dots, x_n)$ to be moved to build a FORTIFICATION: $(y_1, \dots, y_n)$

PROBLEM: minimize

$$\sum_{i=1}^{n} m_i \left| x_i - T(x_i) \right|$$

WEIGHTED TRAVELLED DISTANCE

among bijections $T: \{x_1, \dots, x_n\} \to \{y_1, \dots, y_n\}$

MODERN FORMULATION

Given two probability spaces (X, μ), (Y, ν) find a map $T: X \to Y$ such that

$$T_{\#}\mu = \nu \quad \left(\text{i.e. } \nu(A) = \mu\left(T^{-1}(A)\right) \ \forall A \subseteq Y\right)$$

and T minimizes

$$\int_X c(x, T(x)) \, d\mu$$

where $c: X \times Y \to \mathbb{R}$ is a given COST

BRANCHED TRANSPORT

OBJECTIONS TO THE PREVIOUS MODEL:

- transport is DYNAMIC, the model is STATIC.
- the cost does not take into account:
  - trajectories of moving particles,
  - interactions between particles.

MAIN FEATURE: we search for a model which favors LARGE FLOWS, i.e. it is more efficient to transport goods in a GROUPED WAY rather than in a SEPARATE WAY

EXAMPLES:

- blood vessels / nervous system
- nerves of a leaf / tree roots & branches
- drainage networks / river basins
- power supply / distribution of water
- ground transportation

THE DISCRETE MODEL

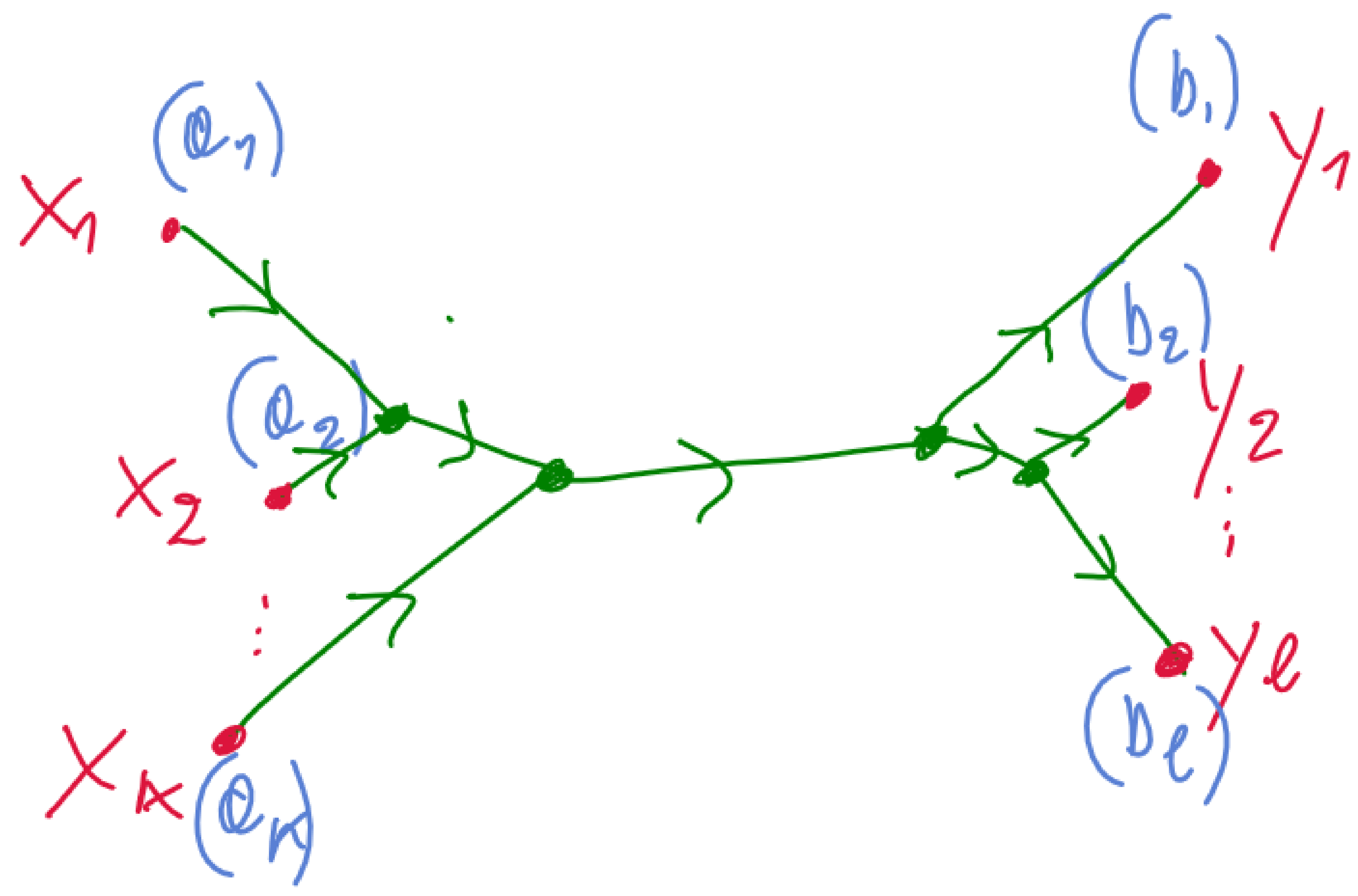
(E.N. GILBERT, Minimum cost communication networks, 1967)

- the INITIAL MEASURE μ^- and the TARGET MEASURE μ^+ are FINITE ATOMIC

$$\mu^- := \sum_{i=1}^{k} a_i \delta_{x_i} \qquad \mu^+ := \sum_{j=1}^{\ell} b_j \delta_{y_j}$$

- the transportation network is represented by a finite ORIENTED WEIGHTED GRAPH (G, w) where the function $w: E(G) \to \mathbb{R}$ represents the flow of mass and satisfies the KIRCHHOFF'S LAWS

"disjoint"



at x_i: $\displaystyle\sum_{e:\text{ OUTGOING}} w(e) - \sum_{e:\text{ INCOMING}} w(e) = a_i$

at y_j: $\displaystyle-\sum_{e:\text{ OUTGOING}} w(e) + \sum_{e:\text{ INCOMING}} w(e) = b_j$

at every other node: $\displaystyle\sum_{e:\text{ OUTGOING}} w(e) = \sum_{e:\text{ INCOMING}} w(e)$

- We quotient oriented weighted graphs w.r.t. the equivalence relation:

$$(w)\nearrow_e \sim (-w)\swarrow_e$$

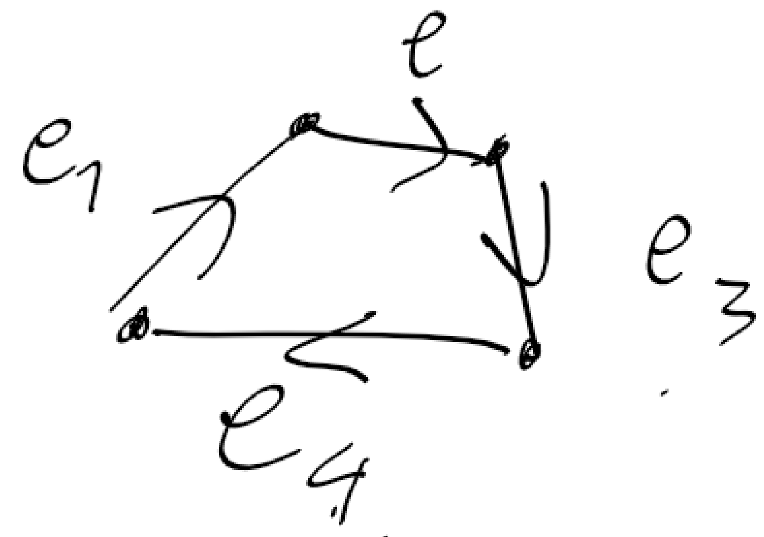
- Fix $\alpha \in [0,1]$.
Among all oriented weighted graphs "transporting" μ^- onto μ^+ we seek a minimizer G of the α-ENERGY

$$\mathbb{E}^\alpha(G) := \sum_{e \in E(G)} |w(e)|^\alpha \cdot \mathcal{H}^1(e)$$

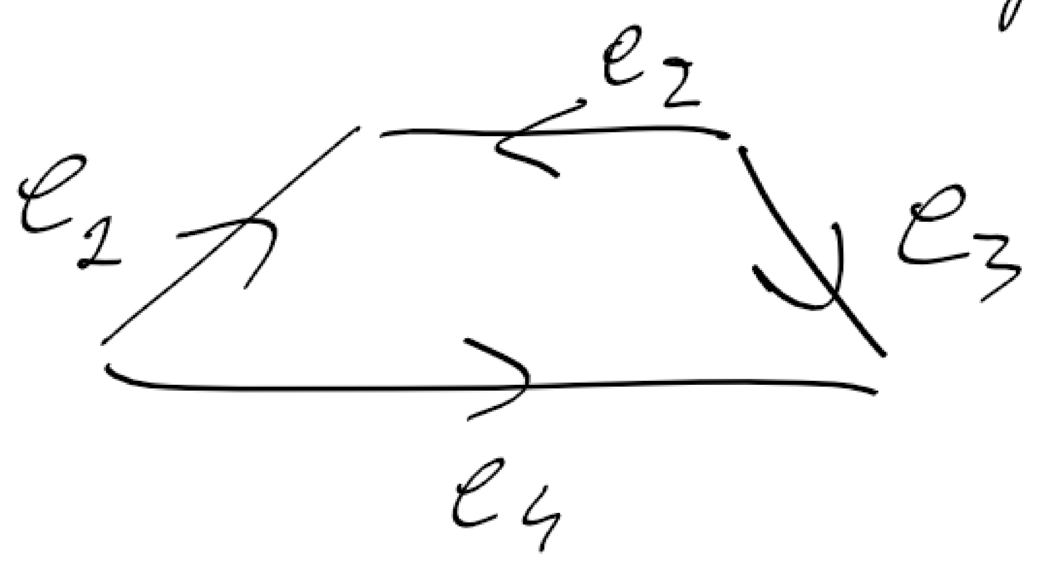
BASIC PROPERTIES OF MINIMIZERS

NOTATION:

- A CYCLE is a closed chain $(e_1, \dots, e_n)$ of (consecutive) distinct edges

(CYCLE) 

- A LOOP is a chain of edges which can be turned into a cycle possibly switching the orientation of some edges.

(LOOP)
(NOT A CYCLE) 

- We will consider graphs with positive weights
- An oriented weighted graph is ACYCLIC if it does not contain any cycle.

It is a TREE if it does not contain any loop.

PROPOSITION Let (G, w) be a minimizer of $\mathbb{E}^\alpha$. Then G is acyclic and moreover it is a tree.

proof: (i) G is acyclic: subtracting a cycle reduces $\mathbb{E}^\alpha$ by monotonicity of

$x \mapsto x^\alpha$

with constant weight

... proof (ii) G is a tree:

- Assume by contrad. that G contains a loop $L \subseteq E(G)$.
- Consider a cycle L' obtained by switching the orientations on a subset L^- of L and let $L^+ := L \setminus L^-$

- Assume without loss of generality that

$$(1) \qquad \sum_{e \in L^+} \alpha\, w(e)^{\alpha-1} \cdot \mathcal{M}'(e) \leq \sum_{e \in L^-} \alpha\, w(e)^{\alpha-1} \cdot \mathcal{M}'(e)$$

(otherwise switch all orientations in $E(G)$)

- Observe that, by concavity of $x \to x^\alpha$, $\forall\, w > 0$, $\forall\, 0 < \varepsilon \leq w$:

$$(2) \qquad (w \pm \varepsilon)^\alpha < w^\alpha \pm \alpha\, w^{\alpha-1} \cdot \varepsilon$$

- Let ε be the smallest (positive) weight in L and let L'' be the graph with $E(L'') = L'$ and the weight is ε on each edge.

- We estimate:

$$\mathbb{E}^\alpha(G + L'') - \mathbb{E}^\alpha(G) = \sum_{e \in L^+} (w(e)+\varepsilon)^\alpha \mathcal{M}'(e) + \sum_{e \in L^-} (w(e)-\varepsilon)^\alpha \cdot \mathcal{M}'(e) - \sum_{e \in L} w(e)^\alpha \cdot \mathcal{M}'(e)$$

$$\underset{(2)}{\leq} \varepsilon \sum_{e \in L^+} \alpha\, w(e)^{\alpha-1} \cdot \mathcal{M}'(e) - \varepsilon \sum_{e \in L^-} \alpha\, w(e)^{\alpha-1} \cdot \mathcal{M}'(e) \underset{(1)}{\leq} 0$$

- This contradicts the minimality of G, because $G + L''$ also transports μ^- onto μ^+. $\square$

VARIATIONAL FORMULATIONS

① EULERIAN FORMULATION

(Xia: Optimal paths related to transport problems, 2003)

- TRANSPORTATION NETWORK: (TRANSPORT PATH)
  1-dimensional normal current T on $\mathbb{R}^d$ with $\partial T = \mu^+ - \mu^-$
  (equivalently: vector-measure T with $\operatorname{div} T = \mu^- - \mu^+$)

- COST FUNCTIONAL: (α-MASS)
  Lower semicontinuous relaxation of Gilbert's energy $\mathbb{E}^\alpha$ with respect to the FLAT-CONVERGENCE

It turns out that the relaxation coincides with

$$\boxed{M^\alpha(T) = \begin{cases} \displaystyle\int_E \theta^\alpha \, d\mathcal{H}^1 & \text{if } T = [E, \tau, \theta] \text{ is RECTIFIABLE}^* \\ +\infty & \text{otherwise} \end{cases}}$$

(STATIC MODEL)

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* Namely the associated vector-measure has the form

$$T = \vec{\tau} \otimes \underline{\theta \cdot \mathcal{H}^1 \llcorner E} \quad \text{with}$$

- E : 1-RECTIFIABLE SET
- $\vec{\tau} \in \operatorname{Tan}(E)$, $|\tau| = 1$ $\mathcal{H}^1$-a.e.
- $\theta \in L^1(\mathcal{H}^1 \llcorner E)$, $\theta \geq 0$

② LAGRANGIAN FORMULATION

(F. Maddalena, S. Solimini, J.M. Morel: A variational model of irrigation patterns, 2003)

- TRANSPORTATION NETWORK: (TRAFFIC PLAN)

Measure $\mathbb{P}$ on the space $Lip_1(\mathbb{R}^+)$ of 1-Lipschitz paths $\gamma : [0,+\infty) \mapsto \mathbb{R}^d$, supported on the subset of paths having finite length. (+ ...)

- CONSTRAINT denote e_0 and e_∞ the maps : $Lip_1(\mathbb{R}^+) \to \mathbb{R}^d$.

$$e_0(\gamma) := \gamma(0) \qquad e_\infty(\gamma) := \lim_{t\to\infty} \gamma(t).$$

We say that $\mathbb{P}$ is a traffic plan between μ^- and μ^+ provided

$$e_{0\#}\mathbb{P} = \mu^- \quad \text{and} \quad e_{\infty\#}\mathbb{P} = \mu^+$$

- COST FUNCTIONAL (α-ENERGY)

for every $x \in \mathbb{R}^d$ denote its MULTIPLICITY

$$\theta_{\mathbb{P}}(x) := \mathbb{P}\left(\{\gamma \in Lip_1(\mathbb{R}^+) : x \in Im\,\gamma\}\right)$$

Denote the α-energy of $\mathbb{P}$ by

$$\boxed{E^\alpha(\mathbb{P}) := \int_{Lip_1(\mathbb{R}^+)} \int_0^\infty \theta_{\mathbb{P}}(\gamma(t))^{\alpha-1} \, |\dot\gamma(t)| \, dt \, d\mathbb{P}(\gamma)}$$

with the convention that $0^{\alpha-1} = +\infty$.

(DYNAMICAL MODEL - STATIC COST)

3) SYNCHRONIZED TRAFFIC PLANS

(M. Bernot, A. Figalli: Synchronized traffic plans and stability of optima, 2008)

- TRANSPORTATION NETWORK : traffic plan, as in 2)
- COST FUNCTIONAL (α-COST)

Define a modified MULTIPLICITY

$$\tilde{\theta}_{\mathbb{P}}(x,t) = \mathbb{P}\{\gamma \in \mathrm{Lip}_1(\mathbb{R}^+) : \gamma(t) = x\}$$

Denote the α-cost of $\mathbb{P}$ by

$$\boxed{C^\alpha(\mathbb{P}) := \int_{\mathrm{Lip}_1(\mathbb{R}^+)} \int_0^\infty \tilde{\theta}_{\mathbb{P}}(\gamma(t),t)^{\alpha-1} \cdot |\dot{\gamma}(t)|\, dt \, d\mathbb{P}(\gamma)}$$

4) BENAMOU-BRENIER APPROACH

(A. Brancolini, G. Buttazzo, F. Santambrogio: Path functionals over Wasserstein spaces, 2008)

- TRANSPORTATION NETWORK: Lipschitz path Γ in a Wasserstein space W_p of probability measures s.t. $\Gamma(0) = \mu^-$ and $\Gamma(1) = \mu^+$
  - COST FUNCTIONAL
    - For a measure μ define the functional

$$J(\mu) := \begin{cases} \sum_{k\in\mathbb{N}} (a_k)^\alpha & \text{if } \mu = \sum_{k\in\mathbb{N}} a_k \delta_{x_k} \\ +\infty & \text{otherwise} \end{cases}$$

    - For a path Γ as above define

$$\boxed{J^\alpha(\Gamma) := \int_0^1 J(\Gamma(t)) \cdot |\Gamma'|(t)\, dt}$$

where $|\Gamma'|$ is the metric derivative of Γ in the Wasserstein space W_p

(5) MAILING PROBLEM

TRANSPORTATION NETWORK: traffic plan, as in (2)

COST FUNCTIONAL: α-ENERGY as in (2)
or α-COST as in (3)

CONSTRAINT:

- denote $\pi_1, \pi_2 : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$ the projection on the first and second factor, respectively.

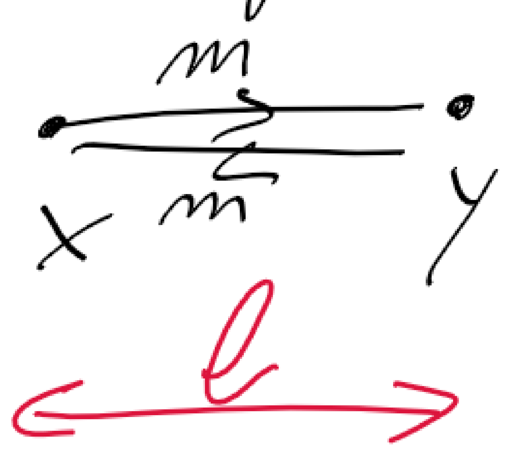
- fix a COUPLING between μ^- and μ^+, namely a measure σ on $\mathbb{R}^d \times \mathbb{R}^d$ such that

$$\begin{cases} \pi_{1\#}\sigma = \mu^- \\ \pi_{2\#}\sigma = \mu^+ \end{cases}$$

- the minimization is among all traffic plans $\mathbb{P}$ such that
$$(e_0, e_\infty)_\# \mathbb{P} = \sigma$$

- In other words for each particle the target point is already established and the trajectory is the only unknown

REMARK: if $\mu^+ = \mu^- = m\delta_x + m\delta_y$ and $\sigma = m\delta_{(x,y)} + m\delta_{(y,x)}$ the minimizer both for α-COST and α-ENERGY is $\mathbb{P}$:



but $\mathbb{E}^\alpha(\mathbb{P}) = (2m)^\alpha \cdot \ell$
while $\mathbb{C}^\alpha(\mathbb{P}) = 2\, m^\alpha \cdot \ell$

OVERVIEW OF MAIN RESULTS

THEOREM 1: (EXISTENCE OF TRANSPORTS WITH FINITE COST)

Assume that $\alpha > 1 - \frac{1}{d}$. Then for every pair of measures μ^-, μ^+ with compact supports and equal masses. There exist a transport path / traffic plan with finite α-mass / α-energy which transports μ^- onto μ^+.

THEOREM 2 (EQUIVALENCE BETWEEN EULERIAN AND LAGRANGIAN FORMULATIONS)

The problem 1 and the problem 2 are equivalent, namely the minimum is the same and it is possible to construct a minimizer of one problem from a minimizer of the other.

THEOREM 3 (STABILITY OF MINIMIZERS)

For both formulations the minimizers are stable. Namely if T_n are minimizers between μ_n^- and μ_n^+ and $T_n \overset{*}{\rightharpoonup} T$ then T is a minimizer between $\mu^- := \lim_n \mu_n^-$ and $\mu^+ := \lim_n \mu_n^+$.

THEOREM 4 (GENERIC UNIQUENESS OF SOLUTIONS)

For the generic choice of μ^- and μ^+ (in the sense of Baire categories) the minimizer is unique.

THEOREM 5 (INTERIOR REGULARITY OF SOLUTIONS)

Let T be a minimizer transporting μ^- onto μ^+ and let $x \in \mathbb{R}^d \setminus (\mathrm{supp}(\mu^-) \cup \mathrm{supp}(\mu^+))$. Then for every ball $B \ni x$ with $B \cap (\mathrm{supp}(\mu^-) \cup \mathrm{supp}(\mu^+)) = \emptyset$, the restriction of T to B is a finite oriented weighted graph.

INTRODUCTION TO THE THEORY OF CURRENTS

A **K-DIMENSIONAL CURRENT** in $\mathbb{R}^d$ is an element of the dual space of $\mathcal{D}^k(\mathbb{R}^d)$ of smooth k-dimensional forms on $\mathbb{R}^d$ with compact support.

REMARK

The topology considered on $\mathcal{D}^k(\mathbb{R}^d)$ is built in analogy with the topology on test functions for which distributions are the dual space.

NOTATION

The space of k-dimensional currents is denoted $\mathcal{D}_k(\mathbb{R}^d)$.

WEAK* CONVERGENCE A sequence of k-dim. currents T_i weakly* converges to a current T whenever

$$\langle T_i, \omega \rangle \xrightarrow[i\to\infty]{} \langle T, \omega \rangle \qquad \forall\, \omega \in \mathcal{D}^k(\mathbb{R}^d)$$

MASS: The **MASS** of a current $T \in \mathcal{D}_k(\mathbb{R}^d)$ is the quantity

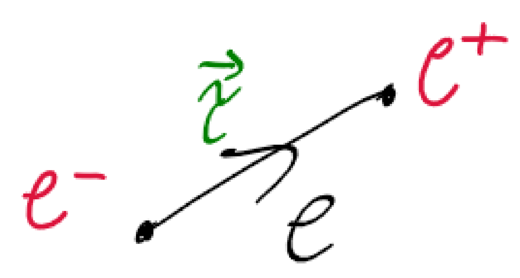
$$\mathbb{M}(T) := \sup \left\{ \langle T, \omega \rangle : \omega \in \mathcal{D}^k(\mathbb{R}^d),\ \|\omega\|_\infty \le 1 \right\}$$

BOUNDARY: The **BOUNDARY** of a current $T \in \mathcal{D}_k(\mathbb{R}^d)$ is the $(k-1)$-dimensional current ∂T satisfying

$$\langle \partial T, \varphi \rangle = \langle T, d\varphi \rangle \qquad \forall\, \varphi \in \mathcal{D}^{k-1}(\mathbb{R}^d).$$

BASIC EXAMPLES:

- **WEIGHTED ORIENTED GRAPHS** Given an oriented line segment e we have



- $\mathbb{M}([\![e]\!]) = \mathcal{H}^1(e)$
- $\partial [\![e]\!] = \delta_{e^+} - \delta_{e^-}$
- $\mathbb{M}(\partial [\![e]\!]) = 2$

$$\langle [\![e]\!], \omega \rangle := \int_e \langle \omega(x), \vec{\tau} \rangle \, d\mathcal{H}^1$$

As a consequence, a weighted oriented graph G transporting a measure μ^- onto a measure μ^+ is a 1-dimensional current with boundary $\partial G = \mu^+ - \mu^-$

- **RECTIFIABLE CURRENTS** A 1-dimensional current is said to be **RECTIFIABLE** if it can be represented by a triple $[E, \tau, \theta]$ where
  - $E \subseteq \mathbb{R}^d$ is a countable union of (supports of) Lipschitz curves
  - $\tau(x) \in Tan(E, x)$ for $\mathcal{H}^1$-a.e. x
  - $\theta \in L^1_{loc}(\mathcal{H}^1 \llcorner E)$

and its action is given by

$$\boxed{\langle [E, \tau, \theta], \omega \rangle := \int_E \langle \omega(x), \tau(x) \rangle \theta(x) \, d\mathcal{H}^1(x)}$$

REMARK The mass of the current $[E, \tau, \theta]$ is

$$\boxed{\mathbb{M}([E, \tau, \theta]) = \int_E |\theta| \, d\mathcal{H}^1}$$

- **ORIENTED SYMPLEXES** Let S be a k-symplex on $\mathbb{R}^d$ (i.e. the convex hull of $k+1$ affinely indep. points) and let σ be an **ORIENTATION** of S. Then we denote by $[\![S]\!]$ the k-dimensional current acting as

$$\langle [\![S]\!] ; \omega \rangle := \int_S \langle \omega(x), \sigma \rangle \, d\mathcal{H}^k(x) \qquad \forall \omega \in D^k(\mathbb{R}^d)$$

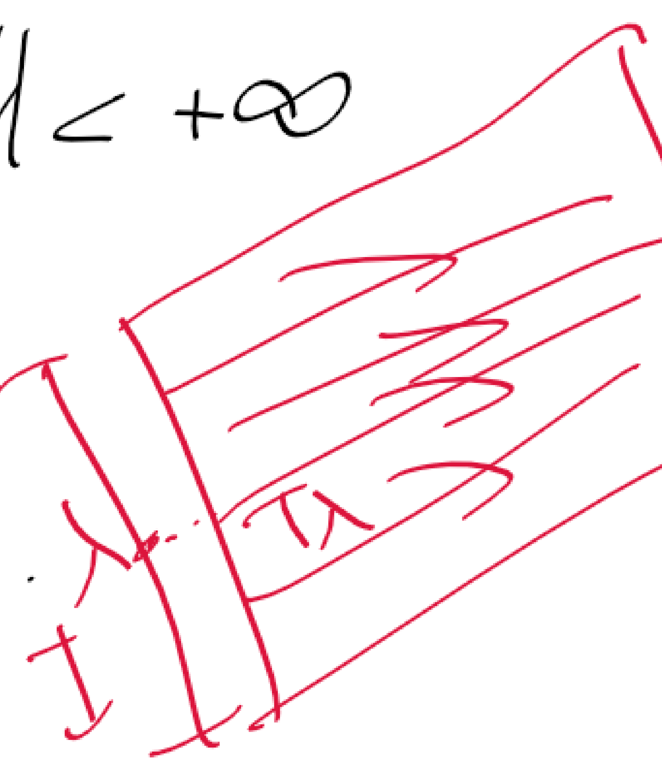
REMARK Similar structure of current can be given to an oriented C^1-surface

- **"INTEGRATION" OF CURRENTS**

Let $(I, d\lambda)$ be a finite measure space and for each $\lambda \in I$ let T_λ be a 1-dimensional current such that $\int \mathbb{M}(T_\lambda) \, d\lambda < +\infty$

We denote $T := \int_I T_\lambda \, d\lambda$ the current defined by

$$\langle T, \omega \rangle = \int_I \langle T_\lambda, \omega \rangle \, d\lambda \qquad \forall \omega \in D^1(\mathbb{R}^d)$$

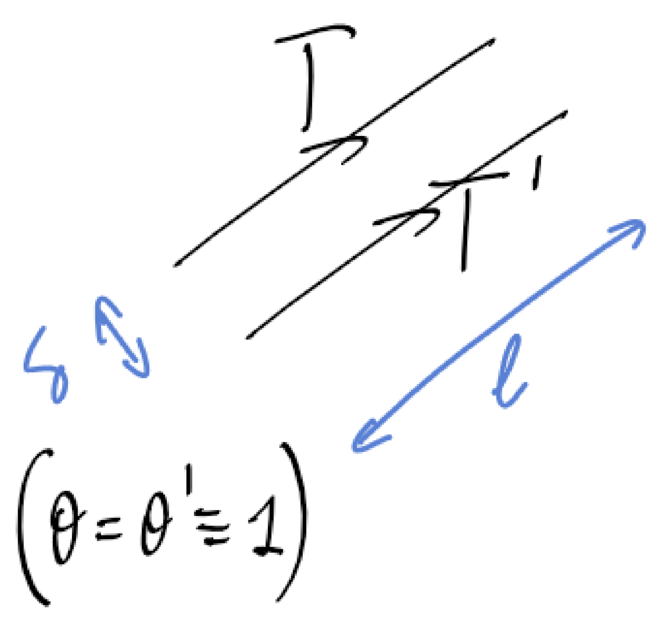


NORMAL CURRENT: current with finite mass whose boundary has finite mass as well

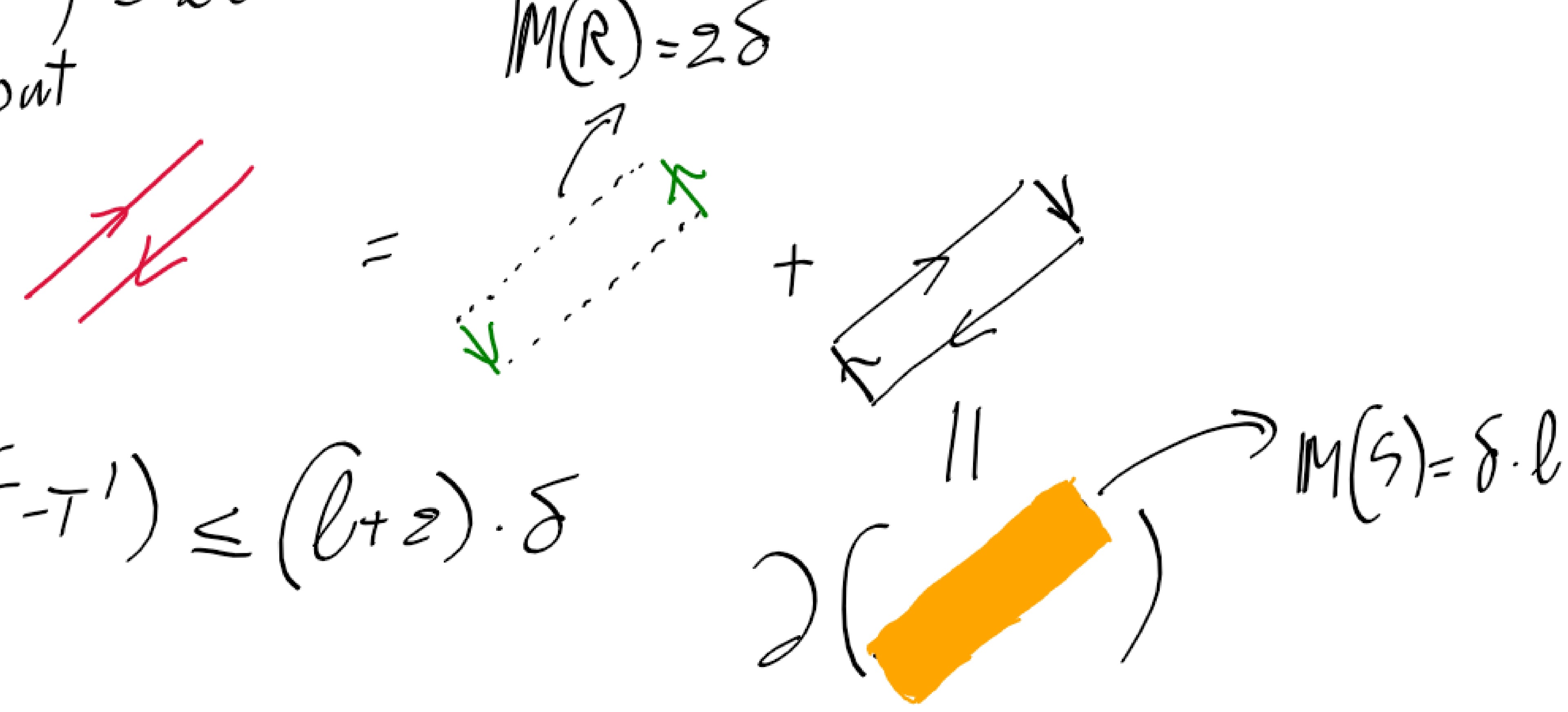
FLAT NORM the FLAT NORM of a current $T \in \mathcal{D}_k(\mathbb{R}^d)$ is the quantity

$$\mathbb{F}(T) := \inf\{\mathbb{M}(R) + \mathbb{M}(S) : T = R + \partial S\}$$

EXAMPLE



• $\mathbb{M}(T - T') = 2\ell$
but
• $T - T' =$



• Hence $\mathbb{F}(T - T') \leq (\ell + 2)\cdot\delta$

POLYHEDRAL CHAIN: finite sum of weighted oriented symplexes

FLAT CHAIN: limit in flat norm of polyhedral chains (see FEDERER 4.2.20)

THEOREM (POLYHEDRAL APPROXIMATION)
Every normal current is a flat chain. Moreover $\forall\, T$ normal $\exists$ Polyhedral chains P_n such that

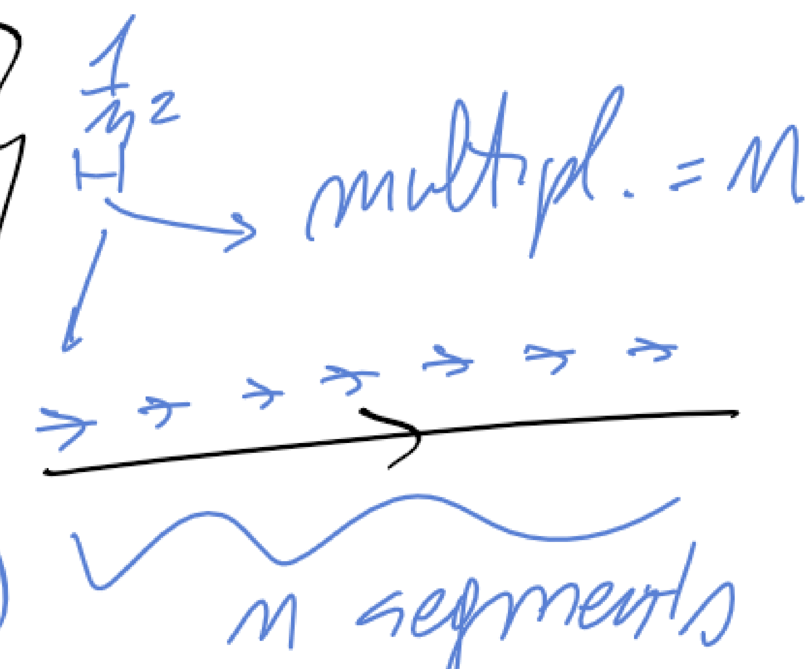
$$\mathbb{F}(P_n - T) \to 0 \qquad \mathbb{M}(P_n) \leq \mathbb{M}(T) \qquad \mathbb{M}(\partial P_n) \leq \mathbb{M}(\partial T)$$

REMARK Convergence in flat norm always implies w^*-convergence
The vice-versa is true under uniform bounds on the masses of the currents and their boundaries.

α-MASS: the α-MASS of a 1-dimensional flat chain T is the quantity

the one defined by Gilbert

$$\mathbb{M}^\alpha(T) := \inf_{\substack{\mathbb{F}(P_n - T) \to 0 \\ (P_n)_n \text{ polyhedral}}} \left\{ \liminf_n \left\{ \mathbb{E}^\alpha(P_n) \right\} \right\}$$



BRANCHED TRANSPORT PROBLEM (EULERIAN FORMULATION)
Among all normal 1-currents T with $\partial T = \mu^+ - \mu^-$, minimize the quantity $\mathbb{M}^\alpha(T)$.

WELL-POSEDNESS FOR $\alpha > 1 - \frac{1}{d}$

From now on we will assume that the ambient space is a convex compact set $X \subseteq \mathbb{R}^d$.

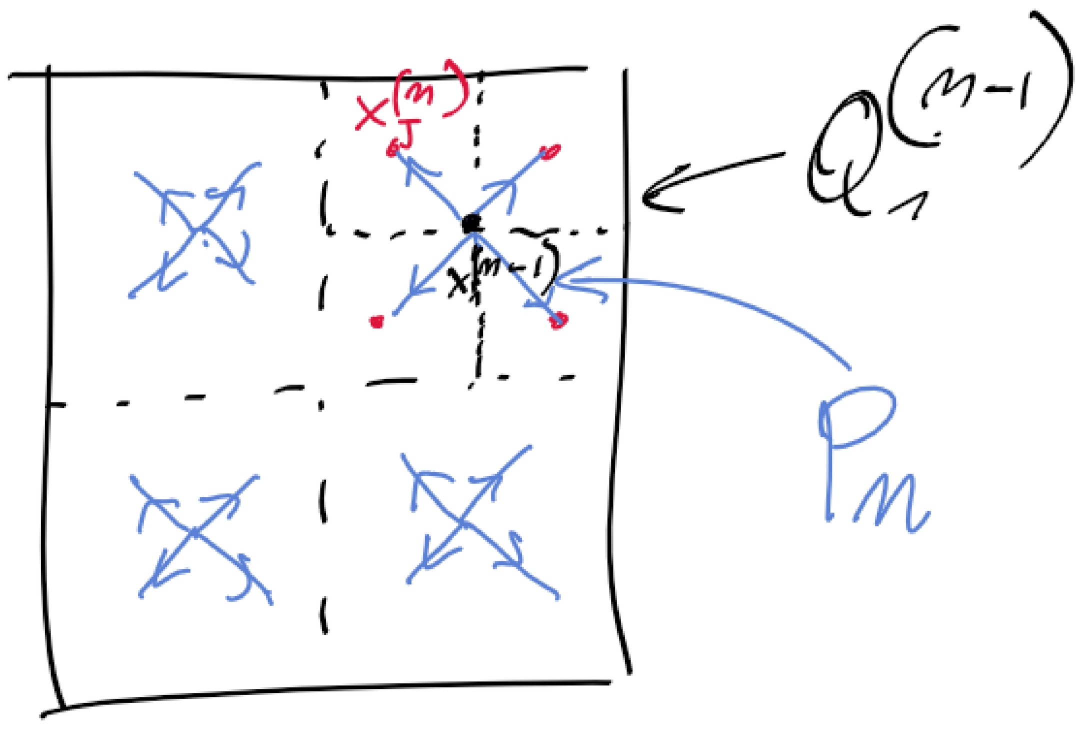
PROPOSITION (EXISTENCE OF TRANSPORTATION NETWORKS WITH FINITE COST) (see XIA 2003 PROPOS. 3.1)

Let $\alpha > 1 - \frac{1}{d}$. Let μ^+, μ^- be measures on X with the same mass. Then there exist a normal 1-current T such that $\partial T = \mu^+ - \mu^-$ and

$$\mathbb{M}^\alpha(T) \le C_{\alpha,d}\, \mathrm{diam}(X)\, \underbrace{\|\mu^\pm\|}_{\text{TOTAL VARIATION}}^{\alpha}$$

proof

- Wlog $X = [-1,1]^d$, $\mu^- = \delta_0$, $\mu^+(\partial Q) = 0$ $\forall$ Q dyadic cube.
- For every n let $\mu_n^+ := \sum_{i=1}^{2^{nd}} a_i^{(n)} \delta_{x_i^{(n)}}$, where
  - $a_i^{(n)} = \mu^+(Q_i^{(n)})$ and $Q_i^{(n)}$ are dyadic cubes of n-th generation
  - $x_i^{(n)}$ is the center of $Q_i^{(n)}$
- Let P_n be the oriented weighted graph transporting μ_{n-1}^+ onto μ_n^+ by connecting each $x_i^{(n-1)}$ to the centers $\{x_j^{(n)}\}_{j=1}^{2^d}$ of the dyadic cubes of the n-th generation contained in $Q_i^{(n-1)}$.
- We have



(1) $$\mathbb{M}(P_n) = \sum_{j=1}^{2^{nd}} C_d\, 2^{-n} a_j^{(n)} \underset{\substack{\uparrow \\ \sum a_j^{(n)} = 1}}{=} C_d \cdot 2^{-n}$$

(2) $$\mathbb{E}^\alpha(P_n) = \sum_{j=1}^{2^{nd}} C_d\, 2^{-n} \left(a_j^{(n)}\right)^\alpha$$

$$= C_d \cdot 2^{-n} \cdot 2^{nd} \left(2^{-nd} \sum_{j=1}^{2^{nd}} \left(a_j^{(n)}\right)^\alpha\right) \underset{\text{JENSEN}}{\le} C_d \cdot 2^{n(d-1)} \Big(2^{-nd} \underbrace{\sum_{j=1}^{2^{nd}} a_j^{(n)}}_{=1}\Big)^\alpha$$

$$= C_d \cdot 2^{n(\underbrace{d-1-d\alpha}_{<0})}$$ because $\alpha > 1 - \frac{1}{d}$

... proof

- By (1) we have that $T_m := \sum_{n=1}^{m} P_n$ is a cauchy in mass (hence flat)
Therefore $\exists$ T flat with $M(T)<\infty$ (M is $\mathbb{F}$-l.s.c.) s.t. $T_m \xrightarrow{\mathbb{F}} T$.
- Moreover $\partial T_m \xrightarrow{\mathbb{F}} \partial T$ hence T transports μ^- onto μ^+

$$\underset{\mu_m^+ - \mu^-}{\partial T_m} \qquad \underset{\mu^+ - \mu^-}{\partial T}$$

- By (2)

$$\mathbb{M}^\alpha(T) \le \liminf_m \{\mathbb{E}^\alpha(T_m)\} \overset{\mathbb{E}^\alpha \text{ is subadditive}}{\le} \lim_m \Big\{\sum_{n=1}^{m} \mathbb{E}^\alpha(P_n)\Big\} \overset{(2)}{\le} C_{\alpha,d} \qquad \square$$

PROPOSITION (REPRESENTATION FORMULA FOR THE α-MASS) (see COLOMBO - DE ROSA - MARCHESE - STUVARD)
Let T be a 1-dimensional flat chain with finite mass. Then

$$\mathbb{M}^\alpha(T) = \begin{cases} \int_E |\theta|^\alpha \, d\mathcal{H}^1 & \text{if } T = [E, \tau, \theta] \text{ is rectifiable} \\ +\infty & \text{otherwise} \end{cases}$$

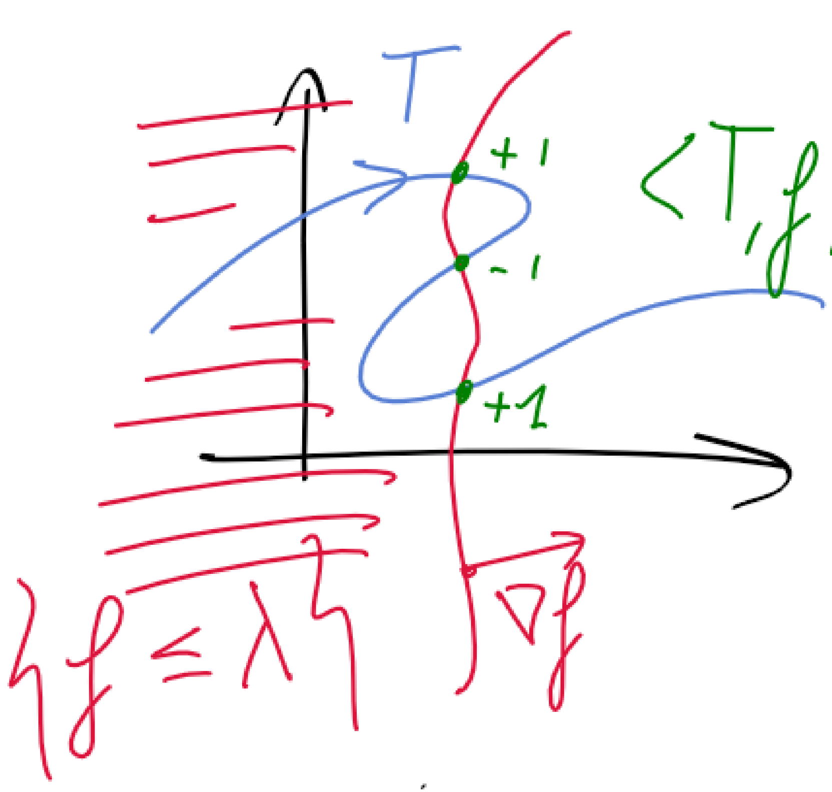
INGREDIENTS FOR THE PROOF

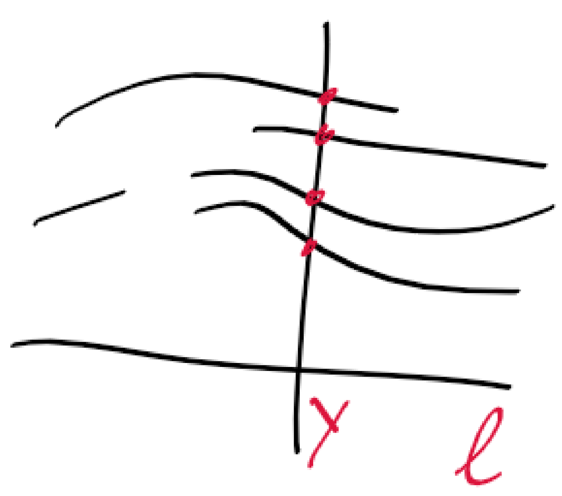
- SLICING OF RECTIFIABLE CURRENTS (See SIMON §28)

Let $T := [E, \tau, \theta]$ be a rectifiable 1-current on $\mathbb{R}^d$ and let $f : \mathbb{R}^d \to \mathbb{R}$ be a lipschitz function. ~~For~~ a.e. $\lambda \in \mathbb{R}$ we define the SLICE of T according to f at level λ as the rectifiable 0-current

$$\langle T, f, \lambda \rangle := [E \cap f^{-1}(\lambda), \tau^\lambda, \theta],$$

where $\tau^\lambda(x) = \begin{cases} +1 & \text{if } \tau(x) \cdot \nabla f(x) > 0 \\ -1 & \text{otherwise} \end{cases}$





Whenever $\mathbb{M}(\partial T) < \infty$ we have the following equality (for a.e. t)

$$\langle T, f, \lambda \rangle = \partial (T \llcorner \{f < \lambda\}) - \partial T \llcorner \{f < \lambda\}$$

restriction

which we take as a definition of slicing when T is normal

Given a line $\ell \subseteq \mathbb{R}^d$ we denote $p_\ell : \mathbb{R}^d \to \ell$ the orth. projection.
We also denote γ the Haar measure on the Grassmannian $Gr(d,1)$

INTEGRALGEOMETRIC FORMULA (see FEDERER 3.2.26)

Let $E \subseteq \mathbb{R}^d$ be 1-rectifiable. There exists $C = C_d$ such that the following equality holds for every Borel function f

$$\int_E f \, d\mathcal{H}^1 = C \int_{Gr(d,1)} \int_\ell \int_{p_\ell^{-1}(\{y\}) \cap E} f(y) \, d\mathcal{H}^0(x) \, d\mathcal{H}^1(y) \, d\gamma(\ell)$$

WHITE'S RECTIFIABILITY CRITERION (see WHITE - RECTIFIABILITY OF FLAT CHAINS - Ann of Math - 1999)

Let T be a 1-dimensional flat chain with finite mass.
If $\langle T, p_\ell, t \rangle$ is 0-rectifiable for a.e. t and for every coordinate line ℓ, then T is a 1-rectifiable current.

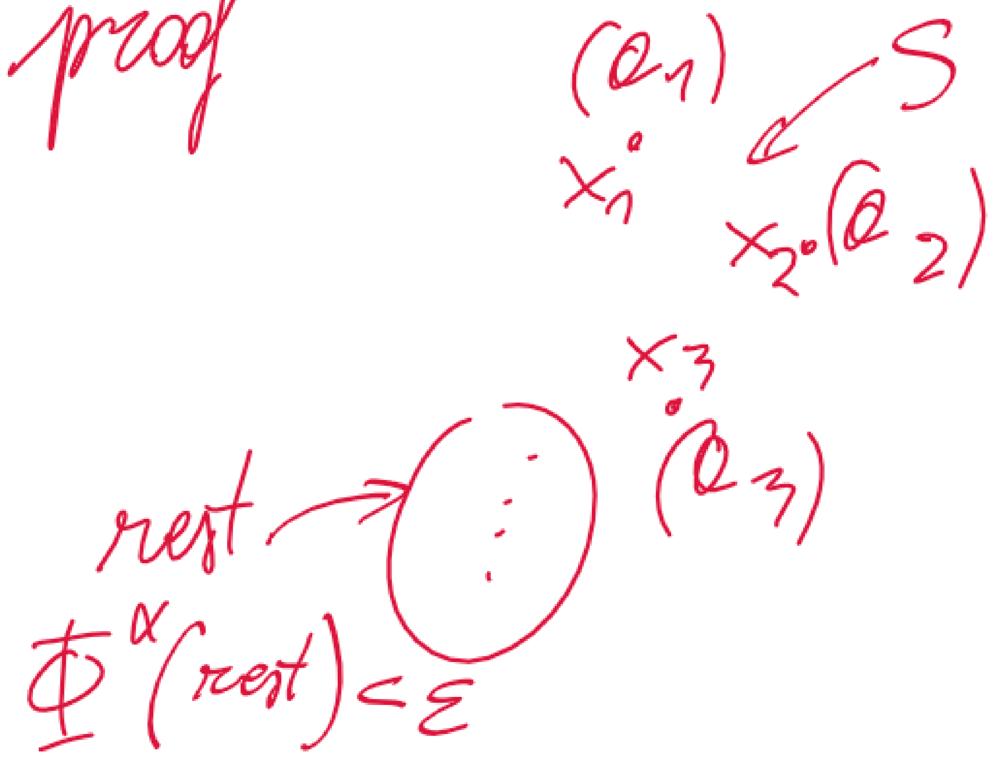
PROOF OF THE PROPOSITION

STEP 1 (SEMICONTINUITY ON RECTIFIABLE 0-CURRENTS)

Let S be a rectifiable 0-current $S = \sum_{i \in \mathbb{N}} a_i \delta_{x_i}$ $(a_i \in \mathbb{R})$
Denote $\Phi^\alpha(S) := \sum_{i \in \mathbb{N}} |a_i|^\alpha$. Then $\forall (R_n)_n$ 0-rectifiable s.t. $\mathbb{F}(R_n - S) \to 0$

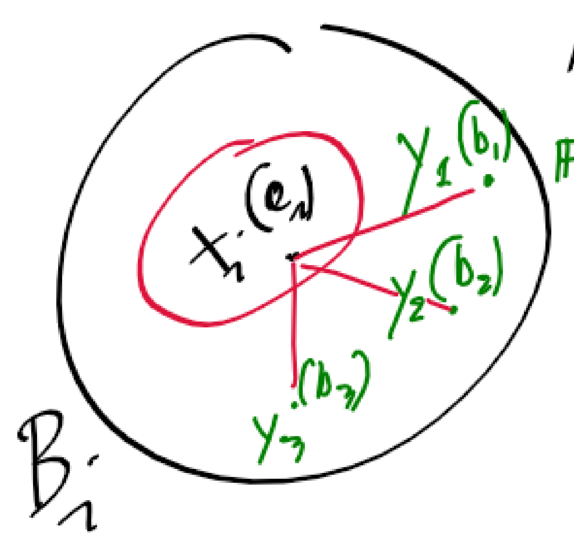
$$\boxed{\Phi^\alpha(S) \le \liminf_{n \to \infty} \{\Phi^\alpha(R_n)\}}$$

proof



- wlog $\Phi^\alpha(S) < +\infty$ (otherwise trivial)
- select n_0 s.t. $\Phi^\alpha\left(\sum_{i > n_0} a_i \delta_{x_i}\right) < \varepsilon$
- cover the remaining points with disjoint balls $\{B_i\}_{i=1}^{n_0}$
- For n large, on each B_i, $\mathbb{F}(S \llcorner B_i - R_n \llcorner B_i) << 1$

hence $R_n \llcorner B_i = \sum_{j \in \mathbb{N}} b_j \delta_{y_j}$ with



$\mathbb{F}$-semicontinuity of $\mathbb{M}$

$$|a_i| < (1+\varepsilon) \sum_{j \in \mathbb{N}} |b_j|$$

subadd. of $x \mapsto x^\alpha$

$$\Downarrow$$

$$|a_i|^\alpha < (1+\varepsilon)^\alpha \sum_{j \in \mathbb{N}} |b_j|^\alpha$$

$\Downarrow$ summing over i

$$\Phi^\alpha(S) - \varepsilon \le (1+\varepsilon)^\alpha \, \Phi^\alpha(R_n).$$

STEP 2: (SEMICONTINUITY ON RECTIFIABLE 1-CURRENTS)

Let $S=[E,\tau,\theta]$ be a rectifiable 1-current and denote $\Phi^\alpha(S):=\int_E |\theta|^\alpha d\mathcal{H}^1$. Then $\forall (R_n)_n$ 1-rectifiable s.t. $\mathbb{F}(R_n-S)\to 0$

$$\boxed{\Phi^\alpha(S)\le \liminf_{n\to\infty}\{\Phi^\alpha(R_n)\}}$$

proof

. From $\mathbb{F}(R_n-S)\to 0$ it follows that, up to subsequences, for γ-a.e. ℓ, for a.e. $y\in\ell$

(the subsequence does NOT depend on ℓ and y)

$$\mathbb{F}(\langle R_n-S, P_\ell, y\rangle)\xrightarrow[n\to\infty]{} 0$$

$\int_\ell \mathbb{F}(\langle T,\pi_\ell,y\rangle)dy \le \mathbb{F}(T)$ (see Federer 4.3.1)

. From STEP 1, for γ-a.e. ℓ, for a.e $y\in\ell$

$$(1)\quad \Phi^\alpha(\langle S,P_\ell,y\rangle)\le \liminf_n\{\Phi^\alpha(\langle R_n,P_\ell,y\rangle)\}$$

. Now from the integralgeometric equality (applied with $f=|\theta|^\alpha$):

$$\Phi^\alpha(S)\underset{\text{integralgeom}}{=}c\int_{Gr(d,1)}\int_\ell \Phi^\alpha(\langle S,P_\ell,y\rangle)dy\, d\gamma(\ell)\underset{(1)}{\le} c\int_{Gr(d,1)}\int_\ell \liminf_n \Phi^\alpha(\langle R_n,P_\ell,y\rangle)dy\, d\gamma(\ell)\underset{\text{FATOU}}{\le}$$

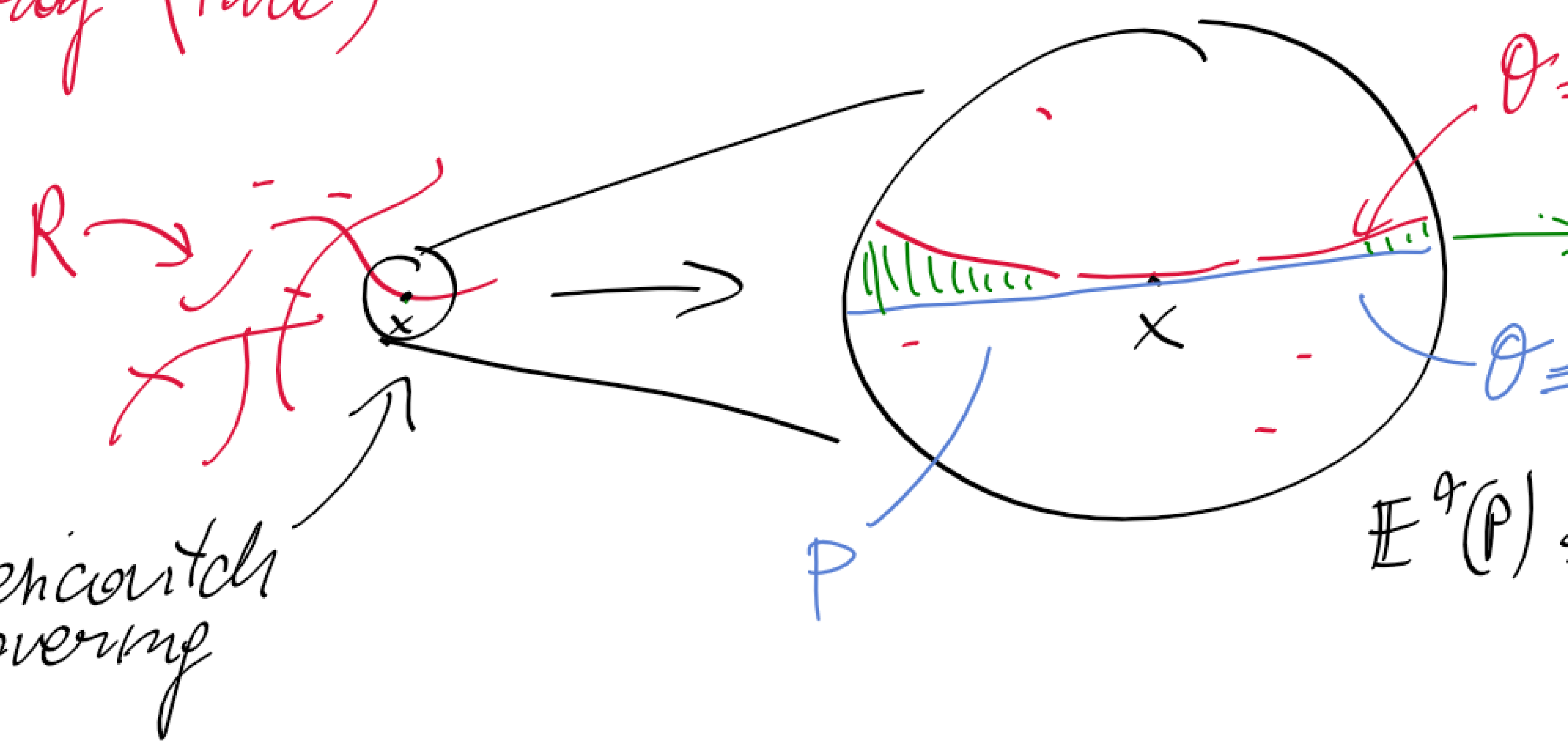
$$c\liminf_n\left\{\int_{Gr(d,1)}\int_\ell \Phi^\alpha(\langle R_n,P_\ell,y\rangle)\,dy\, d\gamma(\ell)\right\}\underset{\text{integralgeom}}{=}\liminf_n\{\Phi^\alpha(R_n)\}.$$

STEP 3 (RECOVERY SEQUENCE FOR Φ^α ON RECTIFIABLE 1-CURRENTS)

Let $R=[E,\tau,\theta]$ be a rectifiable 1-current. For every $\varepsilon>0$ there exists a polyhedral 1-chain P such that:

$$\mathbb{F}(R-P)<\varepsilon,\qquad \mathbb{E}^\alpha(P)\le\Phi^\alpha(R)+\varepsilon$$

proof (idea)



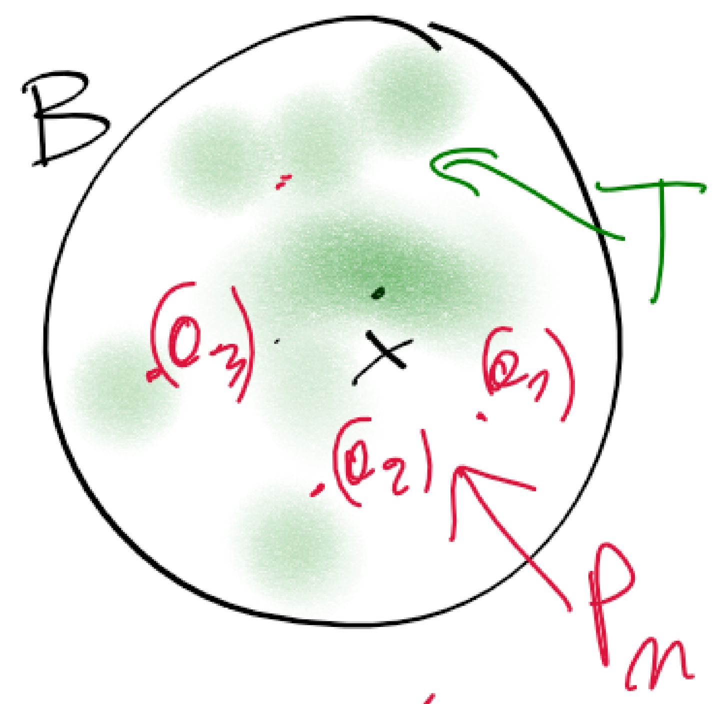
$\mathbb{F}(R-P)$ estimated by homotopy formula (see SIMON 26.23)

$\mathbb{E}^\alpha(P)\le\Phi^\alpha(R)+\varepsilon$

STEP 4 ($\mathbb{M}^\alpha(T) = +\infty$ IF T IS NOT RECTIFIABLE)

proof (idea) for a 0-current T, if T is not rectifiable (i.e. not atomic) then for every sequence P_n of polyhedral 0-chains s.t. $\mathbb{F}(T - P_n) \to 0$, for n large enough we have the following situation at every point $x \in \operatorname{supp}(T_{diff})$ and for every ball B centred at x

diffuse part of T



$\mathbb{F}(P_n - T) \to 0$

$$\sum_i a_i \gtrsim T(B)$$

$$\Phi^\alpha(P_n) >> \sum_i a_i$$

(derivative of x^α $= +\infty$ at 0)

from 0-currents to 1-currents

. Assume by contr. $\exists\, T$ non-rectifiable & $\exists\, (P_n)_n$ polyhedral

$$\mathbb{E}^\alpha(P_n) \le C \qquad \mathbb{F}(P_n - T) \to 0$$

see DE PAUW-HARDT coroll. 3.2.5 (5)

. Then $\int_{V_e} \Phi^\alpha(\langle P_n, p_e, y\rangle)\,dy \le \Phi^\alpha(P_n) \le C \quad \forall\, e$ coordinate line

. Then $\mathbb{M}^\alpha(\langle T, p_e, y\rangle) < +\infty$ for a.e. y, so by 0-dim case, for a.e. y $\langle T, p_e, y\rangle$ is rectifiable $\forall\, e$ coordinate line.

. Then, by White's rectifiability theorem, T is rectifiable $\square$

DEFINITION (ACYCLIC CURRENT)

We say that a current T is ACYCLIC if the only possible S s.t.

$$\partial S = 0 \qquad \mathbb{M}(T) = \mathbb{M}(S) + \mathbb{M}(T - S)$$

is $S = 0$.

PROPOSITION

Let T be a minimizer (with finite α-mass) for a branched transportation problem. Then T is acyclic.

proof

If $T = [E, \tau, \overset{\geq 0}{\theta}]$ and $S = [E', \tau', \overset{\geq 0}{\theta'}]$ the condition $\mathbb{M}(T) = \mathbb{M}(S) + \mathbb{M}(T - S)$ implies that $\mathcal{H}^1$-a.e. on $E \cap E'$ it holds

$$0 \le \theta' \le \theta$$

Hence, as in the discrete case, removing a cycle S reduces $\mathbb{M}^\alpha$ due to the subadditivity of $x \mapsto x^\alpha$ $\square$

SMIRNOV'S THEOREM

(see SMIRNOV / PAOLINI-STEPANOV: "Optimal transportation networks as flat chains" Theorem 6.3)

THEOREM (STRUCTURE OF NORMAL 1-CURRENTS)

Let T be an acyclic normal 1-current with $\mathbb{M}(\partial T) = 2M$. Then there exists a measurable map $I : [0,M] \to Lip_1(\mathbb{R}^+)$

(i) for a.e. $t \in [0,M]$ $I(t)$ is a simple curve with finite length;

(ii) $\int_0^M \mathbb{M}([\![I(t)]\!])\,dt < +\infty$ & $T = \int_0^M [\![I(t)]\!]\,dt$;

$$\Downarrow$$

$$\partial T = \int_0^M \partial [\![I(t)]\!]\,dt$$

(iii) $\mathbb{M}(T) = \int_0^M \mathbb{M}([\![I(t)]\!])\,dt$;

(iv) $\mathbb{M}(\partial T) = \int_0^M \mathbb{M}(\partial [\![I(t)]\!])\,dt$

proof

STEP 1 (DECOMPOSITION OF POLYHEDRAL CHAINS)

Let $P = (E, \tau, \theta)$ be a polyhedral chain with $\theta \in \rho \cdot \mathbb{N}$ for some $\rho > 0$. Then there is a finite atomic measure λ on the space $Lip_1(\mathbb{R}^+)$ (supported on piecewise affine simple curves) such that

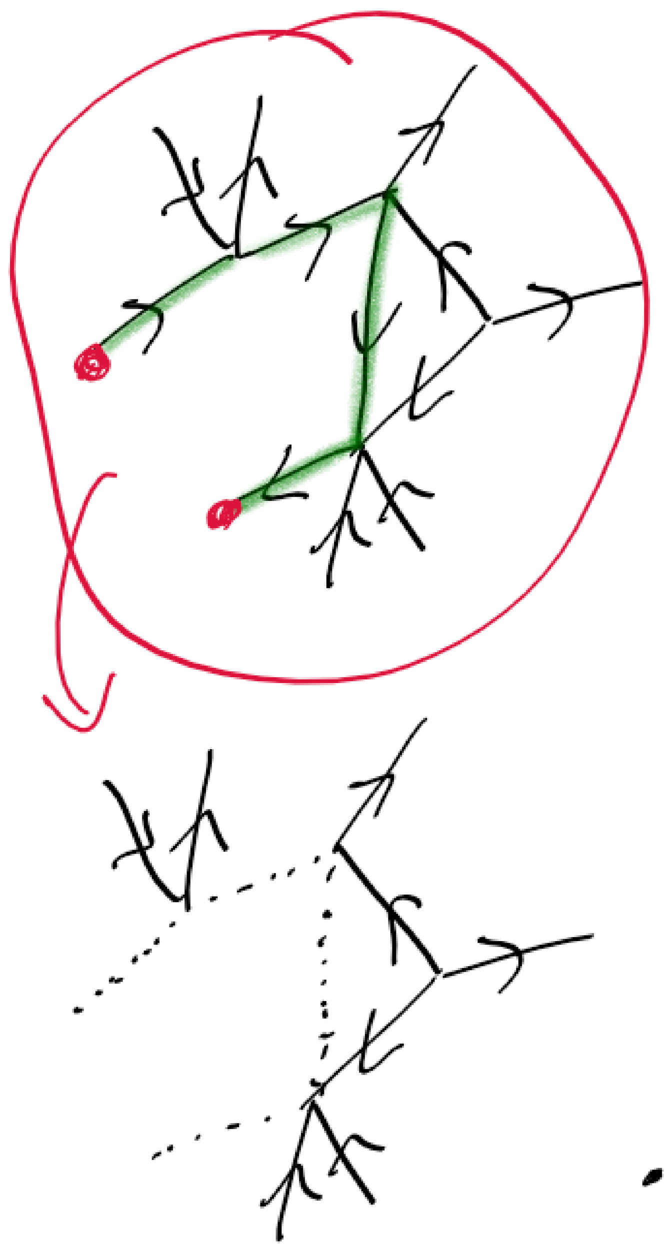
- $P = \int_{Lip_1(\mathbb{R}^+)} [\![\gamma]\!]\,d\lambda(\gamma)$ $\left(\text{Namely } P = \sum_{i=1}^{m} \rho [\![\gamma_i]\!]\right)$

- $\mathbb{M}(P) = \int_{Lip_1(\mathbb{R}^+)} \mathbb{M}([\![\gamma]\!])\,d\lambda(\gamma)$

$$\Longrightarrow \bullet\ \partial P = \int_{Lip_1(\mathbb{R}^+) \cap \{e_0(\gamma) \neq e_\infty(\gamma)\}} \underbrace{\left(\delta_{e_\infty(\gamma)} - \delta_{e_0(\gamma)}\right)}_{= \partial [\![\gamma]\!]}\,d\lambda(\gamma)$$

- $\lambda\left(\underbrace{Lip_1(\mathbb{R}^+) \cap \{e_0(\gamma) \neq e_\infty(\gamma)\}}_{\text{open curves}}\right) = \mathbb{M}(\partial P)$.

PROCEDURE:



- start from a point of ∂P with negative multiplicity and "follow the current" defining a chain of consecutive edges $(e_1, e_2, \dots, e_m)$ until you reach a point with positive multiplicity.
- endow each edge of the chain with multiplicity ρ and remove the current $\rho[\![e_1]\!] + \dots + \rho[\![e_m]\!]$ from P.
- iterate the process until the remaining current has no boundary.
- start from any vertex and define a closed chain
- remove the chain with multiplicity ρ and iterate until all the current is removed.

STEP 2 (DECOMPOSITION OF ACYCLIC NORMAL 1-CURRENTS)

- Let T be an acyclic normal 1-current.
- Apply the polyhedral approximation theorem to obtain a sequence P_n of polyhedral chains such that
  - the multiplicities of P_n belong to $\rho_n \cdot \mathbb{N}$ for some $\rho_n > 0$
  - $\mathbb{F}(P_n - T) \to 0$
  - $\mathbb{M}(P_n) \le \mathbb{M}(T)$
  - $\mathbb{M}(\partial P_n) \le \mathbb{M}(\partial T)$
- Pass to the limit

$$\begin{array}{ccc} P_n & = & \displaystyle\int_{Lip_1(\mathbb{R}^+)} [\![\gamma]\!] \, d\lambda_n(\gamma) \\ \Big\downarrow{\scriptstyle\mathbb{F}} & & \Big\downarrow{\scriptstyle w^*} \\ T & = & \displaystyle\int_{Lip_1(\mathbb{R}^+)} [\![\gamma]\!] \, d\lambda(\gamma) \end{array}$$

- Use the fact that T is acyclic to obtain that λ is supported on simple curves
- By semicontinuity of the mass, obtain (iii) and (iv).
- The existence of the map $I : [0,M] \to Lip_1(\mathbb{R}^+)$ follows from Skorokhod's theorem

CONSEQUENCES OF SMIRNOV'S THEOREM

THEOREM (GOOD DECOMPOSITION OF OPTIMAL TRANSPORT PATHS)

Let T be an acyclic normal 1-current. $2M := \mathbb{M}(\partial T)$ (see COLOMBO - DE ROSA - MARCHESE Th. 2.6)

Then there is a map $I : [0, M] \to Lip_1(\mathbb{R}^+)$ as in Smirnov's theorem with the following additional properties (we denote by $\mathbb{P}$ the measure on $Lip_1(\mathbb{R}^+)$ $\mathbb{P} = I_\# \mathcal{L}^1|_{[0,M]}$)

(1) $\partial_- T = \int_0^M e_{0\#} I(t)\, dt$ & $\partial_+ T = \int_0^M e_{\infty\#} I(t)\, dt$

(2) If $T = [E, \tau, \theta]$ is rectifiable $(\theta \geq 0)$ then, for $\mathcal{H}^1$-a.e. x.

$$\theta(x) = \mathbb{P}(\{\gamma \in Lip_1(\mathbb{R}^+) : x \in Im(\gamma)\})$$

THEOREM (EXISTENCE OF TRANSPORT PATHS MINIMIZING $\mathbb{M}^\alpha$)

Let μ^+, μ^- be such that there exists T with $\partial T = \mu^+ - \mu^-$ and $\mathbb{M}^\alpha(T) < +\infty$.
Then there exists a transport path between μ^- and μ^+ which minimizes the α-mass.

proof

- Let $T_n = [E_n, \tau_n, \theta_n]$ be a minimizing sequence.

(By the representation formula for $\mathbb{M}^\alpha$, T_n are rectifiable)

- By "removing the cycles" from T_n we assume they are acyclic.
  Hence by the previous theorem (point (2)) $|\theta_n| \leq \| \partial_- T_n \| = \mathbb{M}(\mu^-)$.
- By the representation formula for $\mathbb{M}^\alpha$ we have

$$\mathbb{M}(T_n) = \int_{E_n} |\theta_n|\, d\mathcal{H}^1 \leq \frac{\mathbb{M}(\mu^-)}{(\mathbb{M}(\mu^-))^\alpha} \int_{E_n} |\theta_n|^\alpha d\mathcal{H}^1 = \frac{\mathbb{M}(\mu^-)}{\mathbb{M}(\mu^-)^\alpha} \mathbb{M}^\alpha(T_n)$$

Hence $\mathbb{M}(T_n) + \mathbb{M}(\partial T_n) \leq C$

- By the compactness of normal currents, up to subsequence $\exists\, T$ normal s.t. $\partial T = \mu^+ - \mu^-$ and $\mathbb{F}(T_n - T) \to 0$.
- By the l.s.c. of $\mathbb{M}^\alpha$

$$\mathbb{M}^\alpha(T) \leq \liminf_n \mathbb{M}^\alpha(T_n)$$

Hence T minimizes $\mathbb{M}^\alpha$ among normal currents with $\partial T = \mu^+ - \mu^-$.

EQUIVALENCE BETWEEN FORMULATIONS

MAIN FEATURES OF LAGRANGIAN FORMULATION

(see PEBON, §1,2)

- LOWER SEMICONTINUITY OF $\mathbb{E}^\alpha$ ON TRAFFIC PLANS
  - We endow $Lip_1(\mathbb{R}^+)$ with the topology of uniform convergence on compact sets
  - A TRAFFIC PLAN is a finite measure on $Lip_1(\mathbb{R}^+)$ such that

$$\int_{Lip_1(\mathbb{R}^+)} \int_0^\infty |\dot\gamma(t)|\,dt\,d\mathbb{P}(\gamma) < +\infty$$

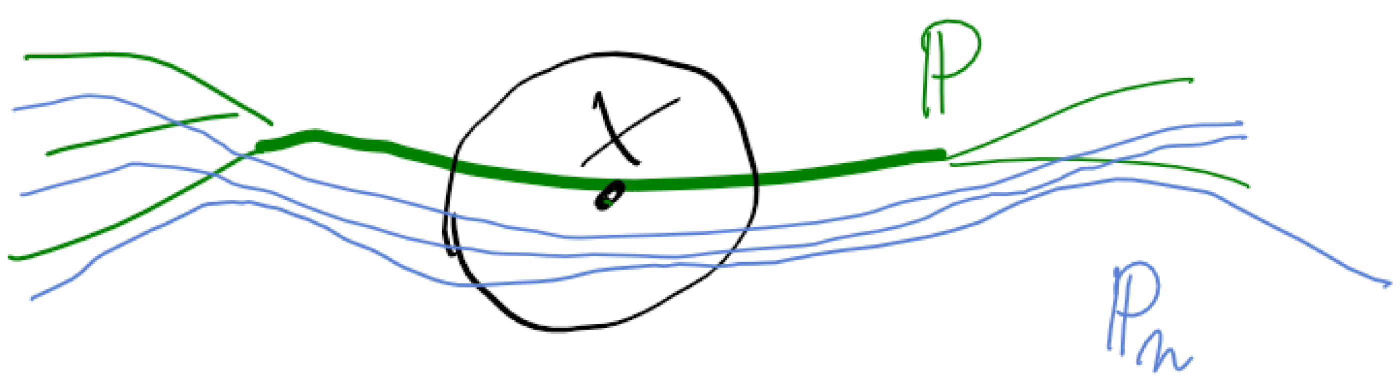
  - If $\gamma \in Lip_1$ is eventually constant, we call $T(\gamma)$ the "stopping time" of γ.
  - For $C>0$, we call TP_C the set of traffic plans $\mathbb{P}$ such that

$$\int_{Lip_1(\mathbb{R}^+)} T(\gamma)\,d\mathbb{P}(\gamma) \leq C \longleftarrow$$

(TIGHTNESS CONDITION: $\forall\, \mathbb{P} \in TP_C$ $\forall \varepsilon>0$, $\mathbb{P}\left(\{\gamma : T(\gamma) \geq \frac{C}{\varepsilon}\}\right) \leq \varepsilon$)

  - For $C>0$ the function

$$\theta : TP_C \times \mathbb{R}^d \to \mathbb{R}^+$$
$$(\mathbb{P}, x) \longmapsto \theta_{\mathbb{P}}(x)$$



is upper semicontinuous

$\Downarrow$

  - For $C>0$ the function $\mathbb{E}^\alpha : TP_C \to \mathbb{R}$ is lower semicontinuous

- EXISTENCE OF SOLUTIONS
  - $\int_{Lip_1(\mathbb{R}^+)} T(\gamma)\,d\mathbb{P}(\gamma) \lesssim \mathbb{E}^\alpha(\mathbb{P})$ hence a minimizing sequence is in TP_C (up to reparametrizations)
  - e_0 and e_∞ are continuous

$\Downarrow$

$$\boxed{\text{there exist a minimizer of } \mathbb{E}^\alpha(\mathbb{P}) \text{ among all traffic plans } \mathbb{P} \text{ s.t. } {e_0}_\# \mathbb{P} = \mu^-,\ {e_\infty}_\# \mathbb{P} = \mu^+.}$$

- PROPERTIES OF THE α-ENERGY

- Denote $\Gamma_{\mathbb{P}} := \{x \in \mathbb{R}^d : \theta(x) > 0\}$. (IF $\mathbb{E}^\alpha(\mathbb{P})$ IS FINITE THEN $\mathbb{P}$-a.e. γ IS SUPPORTED ON $\Gamma_{\mathbb{P}}$)
- If $\mathbb{E}^\alpha(\mathbb{P}) < +\infty$, then $\mathbb{P}$ is rectifiable, namely $\exists$ a rectifiable set $E \subseteq \mathbb{R}^d$ such that for $\mathbb{P}$-a.e $\gamma \in Lip_1(\mathbb{R}^+)$, $Im(\gamma) \underset{\mathcal{H}^1\text{-a.e.}}{\subseteq} E$

(PROOF: SHOW THAT $\Gamma_{\mathbb{P}}$ IS σ-FINITE AND HAS DENSITY 1-a.e.)

- If $\mathbb{P}$-a.e. $\gamma \in Lip_1(\mathbb{R}^+)$ is simple (shortly $\mathbb{P}$ is SIMPLE) and $\mathbb{P}$ is rectifiable, then

$$\mathbb{E}^\alpha(\mathbb{P}) = \int_{\Gamma_{\mathbb{P}}} \theta_{\mathbb{P}}^\alpha(x)\, d\mathcal{H}^1(x)$$

- Optimal traffic plans are SIMPLE
- Optimal traffic plans enjoy the STRONG SINGLE PATH PROPERTY: $\mathbb{P}$-a.e. γ passing through a pair of points (x,y) follow the same trajectory in between (with the same orientation) (similar to Tree property of discrete graphs)

THEOREM (EQUIVALENCE BEETWEEN FORMULATIONS)

Let μ^-, μ^+ be positive measures with the same total variation, then the eulerian and the lagrangian problem are equivalent. Namely, $\forall$ rectifiable transport path $T = [E, \tau, \theta]$ which is acyclic one can find a traffic plan $\mathbb{P}_T$ such that

$\theta \neq 0$

$$\left.\begin{array}{l} \cdot\ \mathcal{H}^1(\Gamma_{\mathbb{P}_T} \Delta E) = 0 \\ \cdot\ \theta_{\mathbb{P}_T}(x) = |\theta(x)| \quad \mathcal{H}^1\text{-a.e.} \\ \cdot\ \mathbb{P}_T \text{ is simple} \end{array}\right\} \quad \text{Hence } \mathbb{E}^\alpha(\mathbb{P}_T) = \mathbb{M}^\alpha(T)$$

Viceversa, $\forall$ rectifiable traffic plan $\mathbb{P}$ which is SIMPLE and satisfies the STRONG SINGLE PATH PROPERTY one can find a transport path $T_{\mathbb{P}} = [E, \tau, \theta]$ such that

$$\left.\begin{array}{l} \cdot\ \mathcal{H}^1(\Gamma_{\mathbb{P}} \Delta E) = 0 \\ \cdot\ \theta_{\mathbb{P}}(x) = |\theta(x)| \quad \mathcal{H}^1\text{-a.e.} \end{array}\right\} \quad \text{Hence } \mathbb{M}^\alpha(T_{\mathbb{P}}) = \mathbb{E}^\alpha(\mathbb{P})$$

In particular, the minimal α-energy and the minimal α-mass coincide *proof*

- Given $T=[E,\tau,\theta]$ acyclic apply Smirnov and write
$$T=\int_0^M [\![I_t]\!]dt \qquad \text{for some } I:[0,M]\to Lip_1(\mathbb{R}^+)$$
and let $\mathbb{P}_T := I_\#(\mathcal{L}^1 \llcorner [0,M])$
By theorem "good decomposition" (2), $|\theta| = \theta_{\mathbb{P}_T}$ for $\mathcal{H}^1$ a.e. $x\in E$
$\mathbb{P}_T$ is simple by Smirnov.

- Given $\mathbb{P}$, let $T_P := \int_{Lip_1} [\![\gamma]\!] \, d\mathbb{P}$
By strong simple path $|\theta| = \theta_{\mathbb{P}}$ for $\mathcal{H}^1$ a.e. $x \in \Gamma_{\mathbb{P}}$ (i.e. there are no "cancellations" at the level of currents) $\square$

STABILITY FOR $\alpha > 1-\frac{1}{d}$

LEMMA Let $\alpha > 1-\frac{1}{d}$. (see BERNOT-CASELLES-MOREL lemma 6.11.)

Let $(\nu^i)_i$, ν be probability measures on X such that
$$\nu^i - \nu \xrightarrow{*} 0.$$
Then $\forall i \; \exists \, T_i$ with $\partial T_i = \nu^i - \nu$ and $\mathbb{M}^\alpha(T_i) \to 0$.

proof

Assume $X = [-1,1]^d$ and w.l.o.g.
$$\nu^i(\partial Q) = \nu(\partial Q) = 0 \quad \forall \text{ dyadic cube } Q.$$
Fix $n\in\mathbb{N}$, and denote $\alpha_1^{(n)}$ the (real valued) measure
$$\alpha_1^{(n)} := \sum_{J=1}^{2^{nd}} a_J^{(n)} \delta_{x_J^{(n)}}$$
where $x_J^{(n)}$ are the centers of the cubes $Q_J^{(n)}$ of the n-th generation
and $a_J^{(n)} := \nu^i(Q_J^{(n)}) - \nu(Q_J^{(n)})$
Notice that $\sum_{J=1}^{2^{nd}} a_J^{(n)} = 0 \quad \forall i, \forall n.$

Since $\nu^i - \nu \overset{*}{\rightharpoonup} 0$ and $\nu(\partial Q)=0$ $\forall Q$ dyadic then

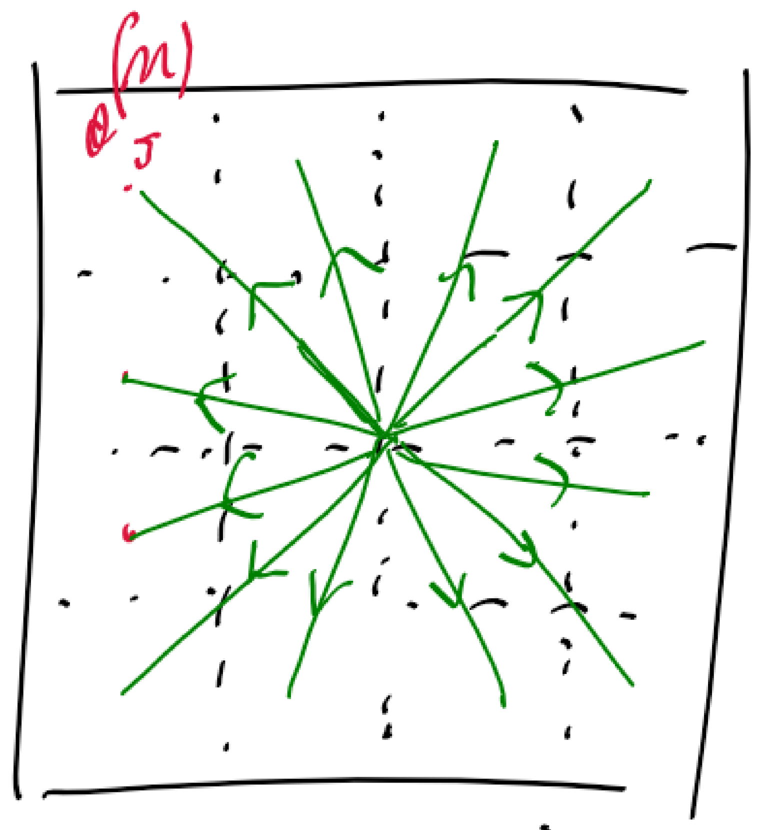
$$a_J^{(n)} \xrightarrow[i\to\infty]{} 0 \quad \forall n, \forall J.$$

In particular $\forall \varepsilon$, $\forall n$ $\exists\, i_0$ s.t. $\forall\, i \geq i_0$

$$(1) \qquad \Phi^\alpha\left(\alpha_i^{(n)}\right) = \sum_{J=1}^{2^{nd}} |a_J^{(n)}|^\alpha < \varepsilon$$

Notice that $\forall i$, $\forall n$, denoting ν_n^i and ν_n the discrete approximations of ν^i and ν respectively (as in the proposition on the existence of transport paths with finite α-mass)

$$\alpha_i^{(n)} = \nu_n^i - \nu_n$$



R_n^i

In particular the polyhedral 1-chain

$$R_n^i := \sum_{J=1}^{2^{nd}} a_J^{(n)} [\![e_J]\!]$$

(where e_J is the oriented segment between 0 and x_J^n), satisfies

$$\partial R_n^i = \nu_n^i - \nu_n \quad \& \quad \mathbb{E}^\alpha(R_n^i) \leq \Phi^\alpha(\alpha_i^{(n)})\cdot C_d \overset{(1)}{<} C_d \cdot \varepsilon$$

On the other hand the proof of the proposition on the existence of transport paths with finite α-mass shows that $\forall \varepsilon$ $\exists\, n_0$ s.t. $\forall\, n \geq n_0$, $\forall i$ $\exists\, S_n, S_n^i$ with

$$\partial S_n = \nu_n - \nu, \quad \partial S_n^i = \nu^i - \nu_n^i, \quad \mathbb{M}^\alpha(S_n) < \varepsilon \quad \mathbb{M}^\alpha(S_n^i) < \varepsilon$$

Hence $T_n^i := S_n^i + R_n^i + S_n$ satisfies

$$\partial T_n^i = \nu^i - \nu, \quad \mathbb{M}^\alpha(T_n^i) < (2 + C_d)\cdot \varepsilon.$$

$\square$

THEOREM (STABILITY OF OPTIMAL TRANSPORT PATHS) (see BERNOT-CASELLES-MOREL: propos. 6.12)

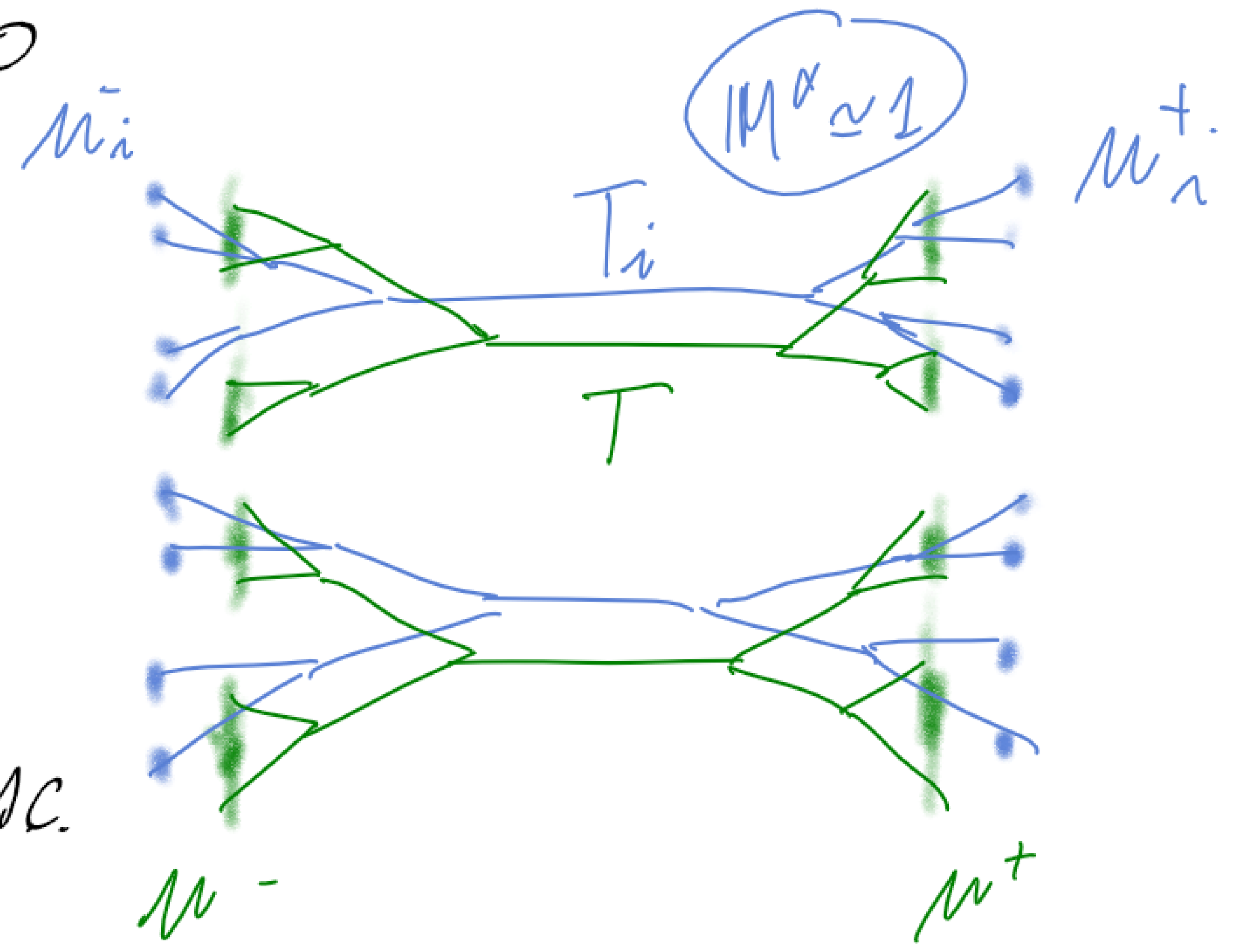
Let $\alpha > 1-\frac{1}{d}$. Assume $(\mu_i^-)_i, (\mu_i^+)_i$ are probability measures s.t. $\mu_i^\pm \overset{*}{\rightharpoonup} \mu^\pm$. Let T_i be optimal transport paths between μ_i^- and μ_i^+ satisfying

$$\sup_i \{\mathbb{M}^\alpha(T_i)\} < +\infty$$

If $\mathbb{F}(T_i - T) \to 0$, then T is an optimal transport path between μ^- and μ^+.

proof

Towards a proof by contradiction, we assume $\exists\, T$ with $\mathbb{F}(T_i - T) \to 0$ and $\exists\, T'$ and $\delta > 0$



$$\partial T' = \partial T = \mu^+ - \mu^-$$

and

(1) $\mathbb{M}^\alpha(T') \leq \mathbb{M}^\alpha(T) - \delta$

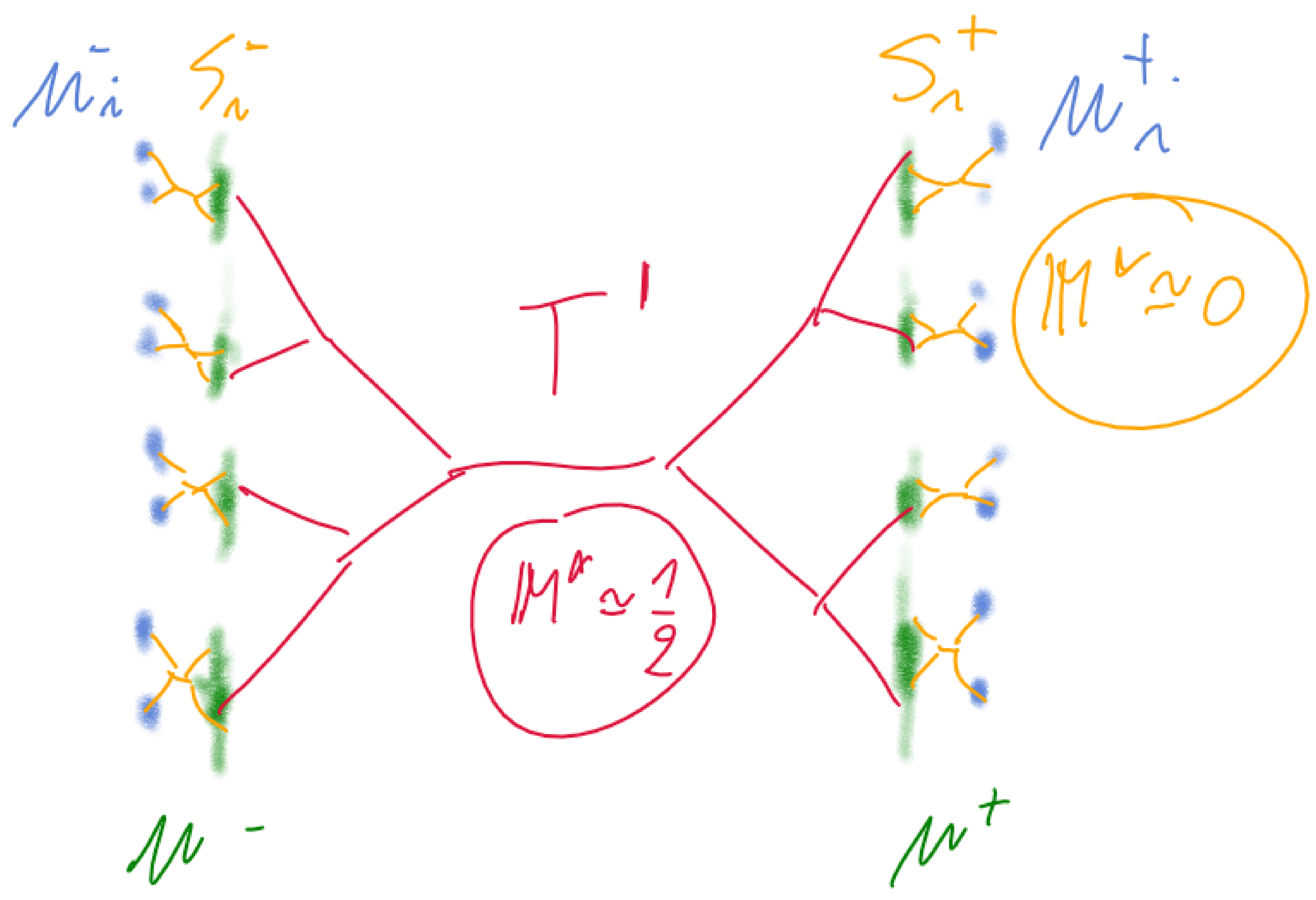
For i sufficiently large we can assume by l.s.c.

(2) $\mathbb{M}^\alpha(T_i) \geq \mathbb{M}^\alpha(T) - \frac{\delta}{4}$

and moreover by lemma we can assume $\exists\, S_i^-, S_i^+$ with

$$\partial S_i^- = \mu^- - \mu_i^- \qquad \partial S_i^+ = \mu_i^+ - \mu^+$$

(3) $\mathbb{M}^\alpha(S_i^\pm) \leq \frac{\delta}{4}$



Hence $R_i := S_i^- + T' + S_i^+$ satisfies

$$\partial R = \mu_i^+ - \mu_i^-$$

so R_i is a competitor for T_i and moreover

$$\mathbb{M}^\alpha(R_i) \leq \mathbb{M}^\alpha(T') + \mathbb{M}^\alpha(S_i^-) + \mathbb{M}^\alpha(S_i^+) \underset{(1),(3)}{\leq} \mathbb{M}^\alpha(T) - \frac{\delta}{2} \underset{(2)}{\leq} \mathbb{M}^\alpha(T_i) - \frac{\delta}{4}$$

which contradicts the minimality of T_i. $\square$

TYPICAL UNIQUENESS OF MINIMIZERS

THEOREM (BAIRE)
Let (M,d) be a complete metric space. Then every countable intersection of open dense sets is dense.

REMARK A set $E \subseteq M$ which contains a countable intersection of open dense subset is called RESIDUAL (equivalently, E is residual if $M \setminus E$ is contained in a countable union of closed sets with empty interior)

NOTATION Fix $C>0$ and denote

$$A_C := \{S : \text{0-current with } \mathbb{M}(S) \le C \ \& \ S = \partial T \text{ for some } T \text{ with } \mathbb{M}^\alpha(T) \le C\}$$

LEMMA A_C is $\mathbb{F}$-closed, hence it is complete

proof Let $(S_i)_i \subseteq A_C$ and let S be s.t. $\mathbb{F}(S_i - S) \to 0$
We want to prove that $S \in A_C$.
By l.s.c. of $\mathbb{M}$, $\mathbb{M}(S) \le C$.
Take T_i s.t. $S_i = \partial T_i$ & $\mathbb{M}^\alpha(T_i) \le C$. Wlog T_i are acyclic, hence $\mathbb{M}(T_i) \le C^{1-\alpha} \mathbb{M}^\alpha(T_i) \le C^{2-\alpha}$
(because the multiplicities θ_i are bounded by C).
By compactness of normal currents $\exists\, T$ with $\mathbb{F}(T_i - T) \to 0$
By continuity of ∂ we have $\partial T = S$ and by l.s.c. of $\mathbb{M}^\alpha$, we have $\mathbb{M}^\alpha(T) \le C$.
Hence $S \in A_C$.

NOTATION We denote by NU_C the subset of A_C

$$NU_C := \{S \in A_C : \exists\, T_1 \ne T_2 \text{ with } T_i \in \operatorname{argmin}\{\mathbb{M}^\alpha(T) : \partial T = S\}\}$$

boundaries with NON-UNIQUENESS of minimizers

LEMMA Assume that $A_C \setminus NU_C$ is dense in A_C. Then it is also residual.
proof For every $K \in \mathbb{N}$ consider the subset of NU_C:

$$NU_C^K = \{S \in A_C : \exists\, T_1, T_2 \in \operatorname{argmin}\{\mathbb{M}^\alpha(T) : \partial T = S\} \text{ with } \mathbb{F}(T_1 - T_2) \ge \tfrac{1}{K}\}$$

We will prove that NU_C^K is closed and with empty interior.

- Since $NU_C^K \subseteq NU_C$, then by assumption $A_C \setminus NU_C^K$ is dense and in particular NU_C^K has empty interior.

... proof

• NU_C^K is closed: let $(S_J)_J \subseteq NU_C^K$ and $S \in A_C$ be s.t. $\mathbb{F}(S_J - S) \to 0$

For every J let

$$\{T_J^{(1)}, T_J^{(2)}\} \subseteq \operatorname{argmin}\{\mathbb{M}^\alpha(T) : \partial T = S_J\}$$

be such that $\mathbb{F}(T_J^{(1)} - T_J^{(2)}) \geq \frac{1}{K}$

As in the previous lemma, we can assume $\exists\, T^{(1)}, T^{(2)}$ such that $\partial T^{(1)} = \partial T^{(2)} = S$ and (up to subs.)

$$\mathbb{F}(T^{(1)} - T_J^{(1)}) \to 0$$
$$\mathbb{F}(T^{(2)} - T_J^{(2)}) \to 0$$

Clearly $\mathbb{F}(T^{(1)} - T^{(2)}) \geq \frac{1}{K}$ and (BY STABILITY)

$$\{T^{(1)}, T^{(2)}\} \subseteq \operatorname{argmin}\{\mathbb{M}^\alpha(T) : \partial T = S\}$$

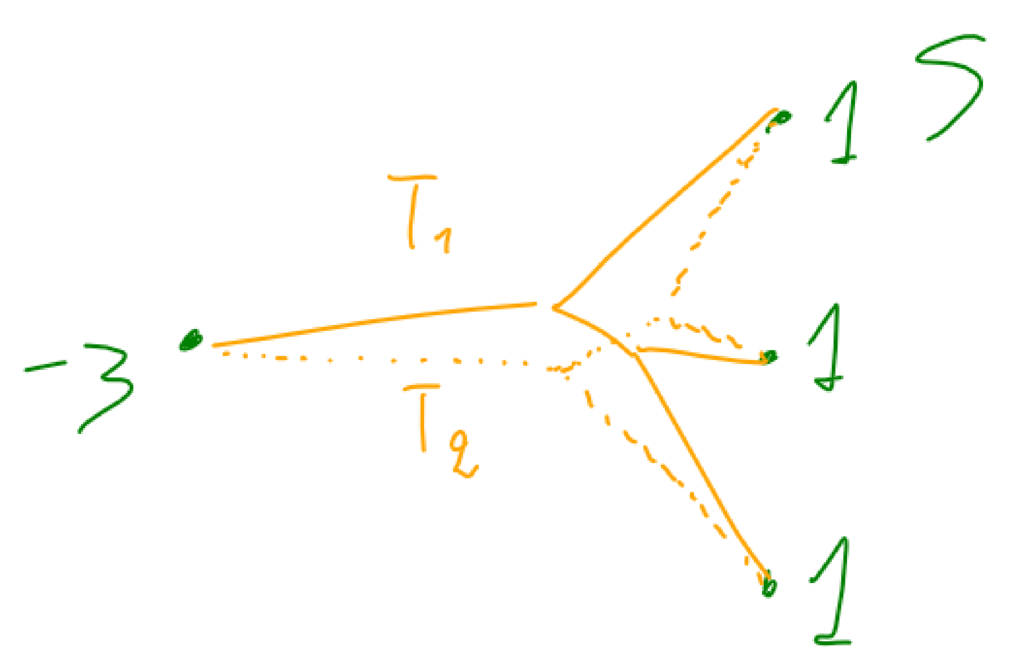
• $NU_C = \bigcup_{K \in \mathbb{N}} NU_C^K$, hence $A_C \setminus NU_C$ is residual.

THEOREM (TYPICAL UNIQUENESS OF MINIMIZERS)

For every $C > 0$, $A_C \setminus NU_C$ is residual.

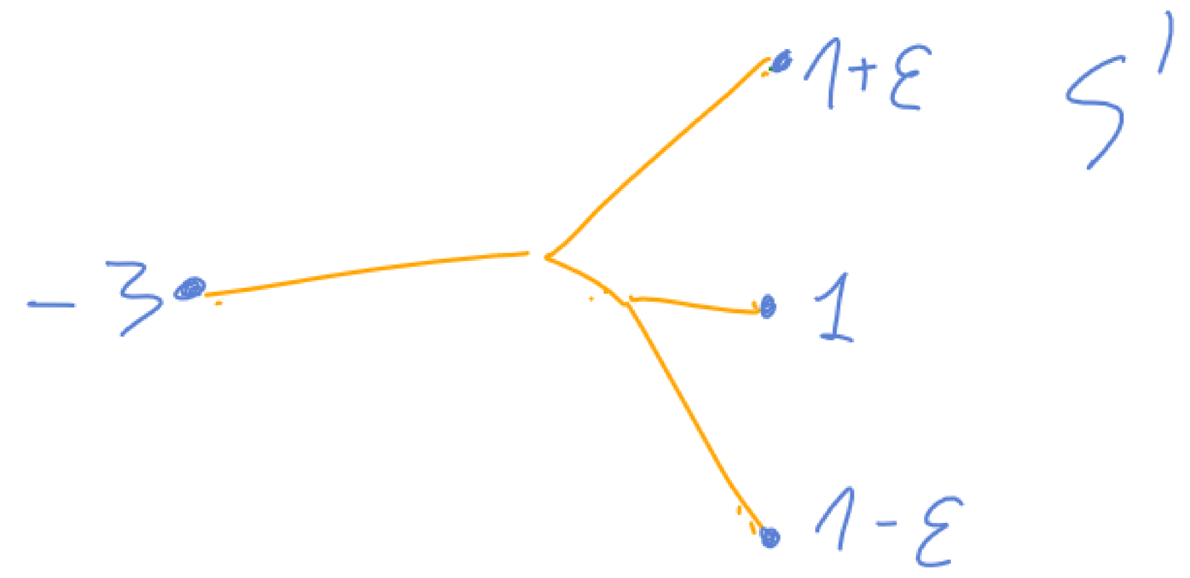
proof

By previous lemma it is sufficient to prove that it is dense.
Let S be a POLYHEDRAL boundary in NU_C. For every $\varepsilon > 0$
We seek for a boundary S' with $\mathbb{F}(S - S') < \varepsilon$ and with
$S' \in A_C \setminus NU_C$.



IDEA:

One can "perturb" S in such a way to select only one of the minimizers.



□

INTERIOR REGULARITY

THEOREM (MONOTONICITY FORMULA) (see XIA - Interior regularity of optimal transport paths - 2004 Corollary 3.1)

Let $T = [E, \tau, \theta]$ be a transport path minimizing $\mathbb{M}^\alpha$. Then the quantity

$$\rho^{-1} \int_{B_\rho(z)} |\theta|^\alpha \, d\mathcal{H}^1 \llcorner E$$

is a nondecreasing function of ρ for $0 < \rho < \operatorname{dist}(z, \partial T)$.

THEOREM (TANGENT CONES)

Let T be as above. Then for every $p \in \operatorname{spt}(T) \setminus \operatorname{spt}(\partial T)$ there exists a tangent cone C_p of T at p.

Moreover (BY STABILITY) C_p minimizes $\mathbb{M}^\alpha$ and this implies that

$$C_p \llcorner B_1 = \sum_{i=1}^{k} m_i [\![e_i]\!]$$

where

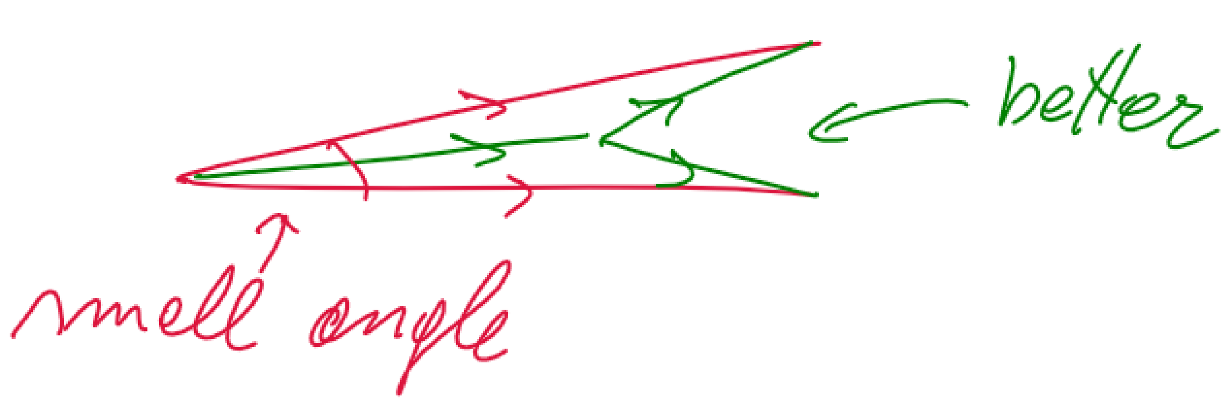
- $k \leq C = C(d, \alpha)$
- e_i are segments of the form $\overline{p_i, 0}$ for some $p_i \in S^{d-1}$
- m_i are real valued multiplicities satisfying

$$\sum_{i=1}^{k} m_i = 0 \qquad \sum_{i=1}^{k} \frac{m_i}{m_i^{1-\alpha}} p_i = 0$$

proof (of the bound on k)

By direct computation one shows that the minimal angle between two vectors e_i and e_j is at least

$\arccos\left(2^{2\alpha - 1} - 1\right)$



$\square$

STABILITY FOR $\alpha \leq 1 - \frac{1}{d}$ (some ideas from the proof)

(see COLOMBO - DE ROSA - MARCHESE On the well-posedness of branched transportation)

LEMMA (A METRIZATION PROPERTY FOR M^α)

Let $Q \subseteq \mathbb{R}^d$ be a cube and $C > 0$. Let $(\mu_n)_n, (\nu_n)_n$ be atomic measures such that

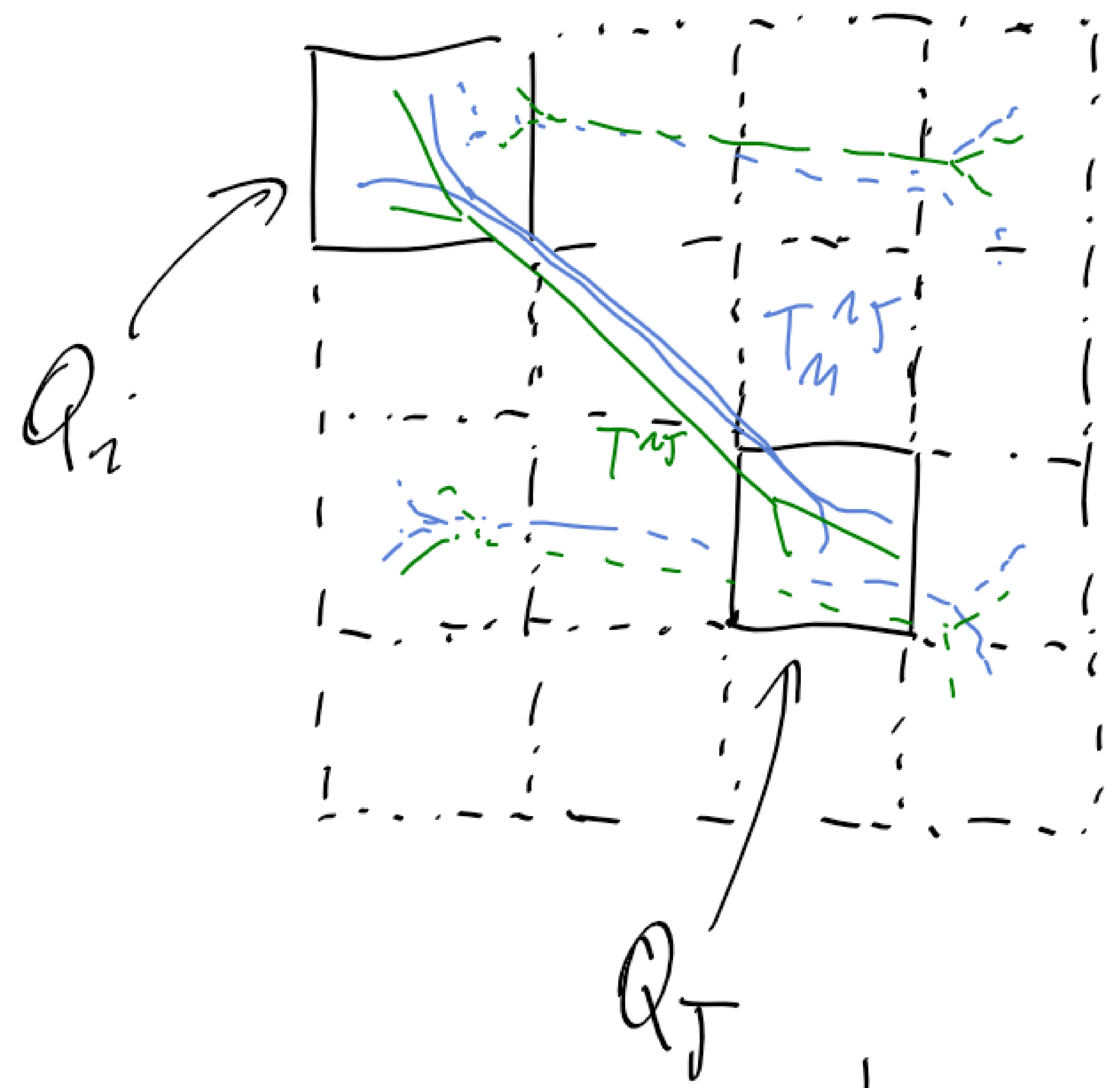
$$\mathbb{M}(\mu_n) = \mathbb{M}(\nu_n), \qquad \mathbb{M}^\alpha(\mu_n) + \mathbb{M}^\alpha(\nu_n) \leq C$$

and $\mu_n - \nu_n \xrightarrow{*} 0$.

Then $\exists\, T_n$ normal with $\partial T_n = \mu_n - \nu_n$ & $\mathbb{M}^\alpha(T_n) \to 0$.

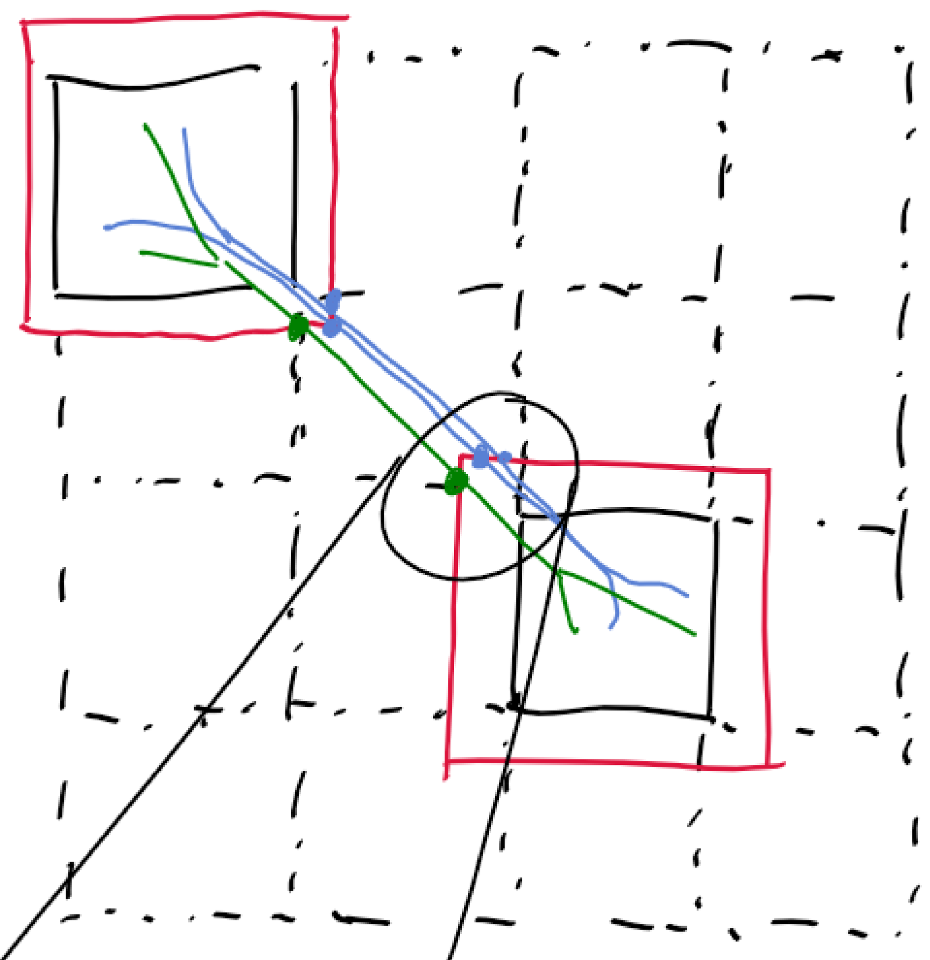
proof similar to the lemma for the stability: the uniform bound on $M^\alpha(\mu_n) + M^\alpha(\nu_n)$ replaces the smallness of the "tails" which was guaranteed by $\alpha > 1 - \frac{1}{d}$.

GOOD DECOMPOSITION OF T AND T_n



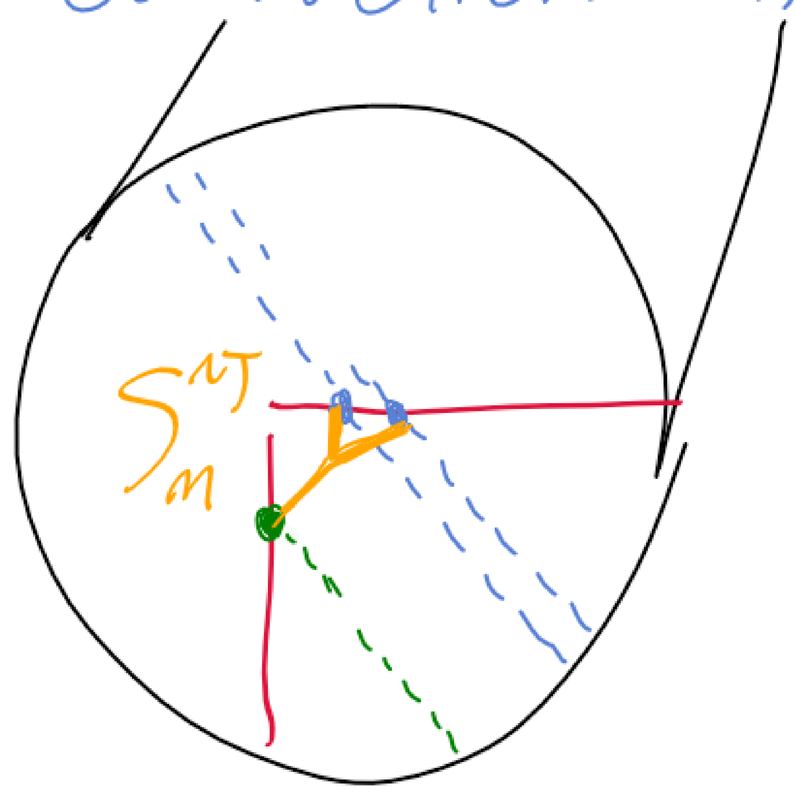
- T is a limit of minimizers T_n
- Decompose T_n with Smirnov and obtain a limit decomposition for T
- $\forall$ pair of dyadic cubes consider T^{ij} and T_n^{ij} to be the part of the current made by paths beginning in Q_i and ending in Q_j

FINDING GOOD SLICES



- Find $\forall (i,j)$ a suitable slice of T_m^{ij} and T^{ij} so that the corresponding slices have equibounded α-mass.

CONNECTION BETWEEN SLICES



- Using the metrization lemma, find a connection S_m^{ij} between the slices.

CONSTRUCTION OF THE COMPETITOR

Assume by contr. $\exists$ T' with $\partial T' = \partial T$ & $E^\alpha(T') \leq E^\alpha(T) - \delta$

Construct a competitor for T_m as follows.

(1) • $\forall (i,j)$ take T_m^{ij} until the slices

(2) • $\forall (i,j)$ connect the slices of T_m^{ij} to the slices of T^{ij} with S_m^{ij}

(3) • $\forall (i,j)$ take T^{ij} (with reversed orientation) from the slices to their boundary inside the cubes

(4) • Take T'

is a competitor for T_m

SOME ISSUES:

(1) does not have necessarily small α-mass (but it is however a piece of T_m)

(3) have small α-mass but the proof is delicate

uses flat chains with coefficients in a group!