Branched covers bounding $\mathbb{Q}$-homology balls

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Project from AIM 2019 meeting on topologically slice knots

Observation 1

- Let $K \subset S^3$ be a knot.
- For $q \in \mathbb{N}$, write $\Sigma_q(K) :=$ q-fold cyclic cover of S^3 branched over K.
- Let $\mathcal{Q} := \{p^r \mid p \in \mathbb{N} \text{ prime}, r \in \mathbb{N}\}$ (prime powers)

Then for $q \in \mathcal{Q}$, $\Sigma_q(K)$ is a $\mathbb{Q}$-homology sphere.

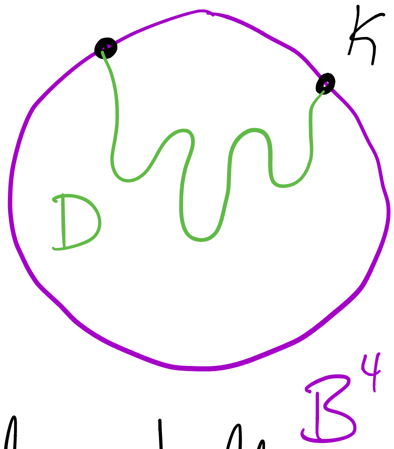
$$(H_*(\Sigma_q(K);\mathbb{Q}) \cong H_*(S^3;\mathbb{Q}))$$

(Classic number theory argument:
$b_1(\Sigma_q K) > 0 \iff \Delta_K(\xi_q) = 0$ for some primitive q-root of unity
(If $q = p^r$) $\iff \Phi_{p^r}(t) \mid \Delta_K(t) \Rightarrow p^r = \Phi_{p^r}(1) \mid \Delta_K(1) = 1$ ※)

Observation 2

- Now take K to be <u>slice</u>

i.e. $K = \partial D \xhookrightarrow{\text{smooth}} B^4$ (disk)



Then for $q \in \mathbb{Q}$,

$\Sigma_q(K)$ bounds a $\mathbb{Q}$-homology ball.

<u>Pf</u> $\Sigma_q(K)$ bounds the q-fold cyclic cover of B^4 branched over D.

$=: W_q(K)$

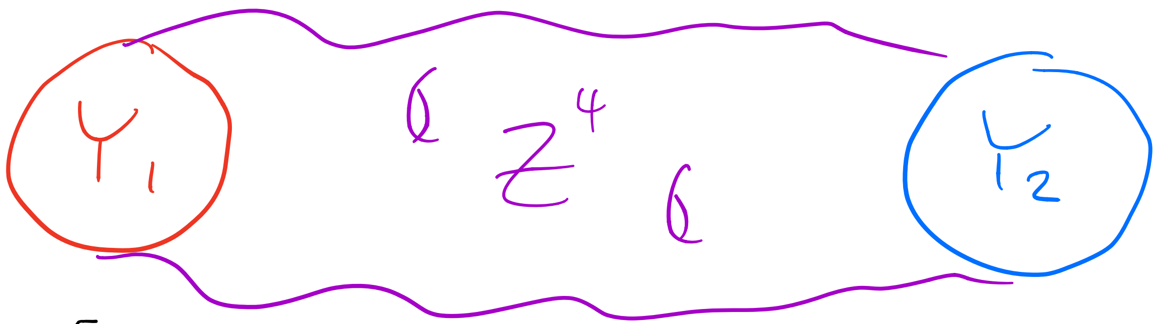
Use similar number theory / Alexander argument to compute $b_1(W_q(K)) = 0$

In fact, $|H_1(W_q K)|^2 = |H_1 \Sigma_q(K)|$, which is useful since $\Rightarrow |H_1 \Sigma_q(K)|$ square

Alternate terminology:

$$[\Sigma_q(K)] = 0 = [S^3] \text{ in } \Theta^3_{\mathbb{Q}}$$

$$\Theta^3_{\mathbb{Q}} = \{\mathbb{Q}HS^3\text{s}\} \big/ \mathbb{Q}\text{-homology cobordism}$$



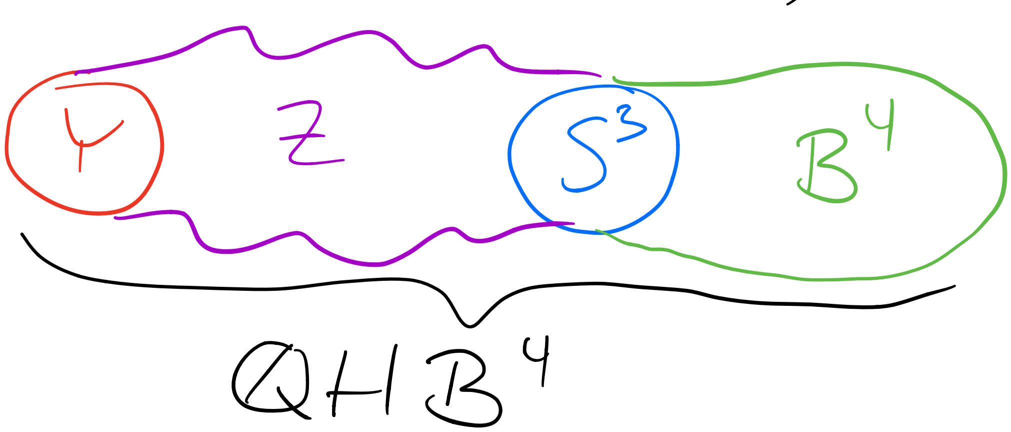
$[Y_1] = [Y_2]$ in $\Theta^3_{\mathbb{Q}}$ if

$\exists\, Z^4$, $\partial\ {}^4 = Y_1 \sqcup -Y_2$,

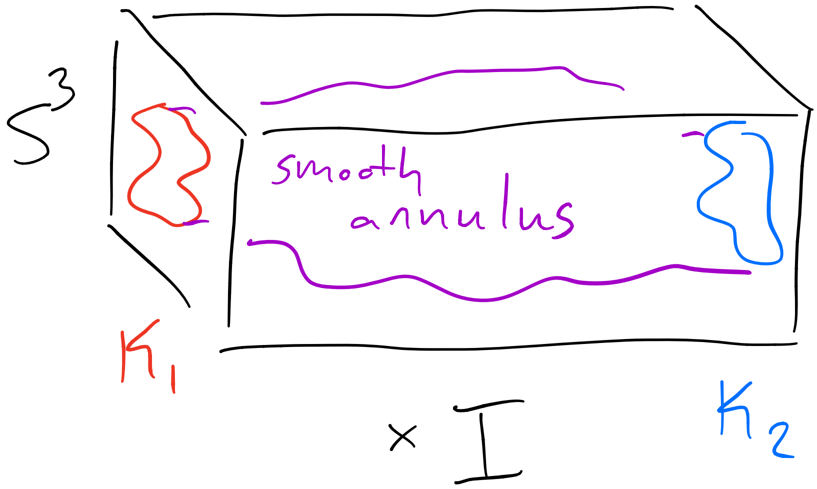
$H_*(Z^4;\mathbb{Q}) \cong H_*(S^3 \times I;\mathbb{Q})$

So $[Y] = [S^3]$ iff

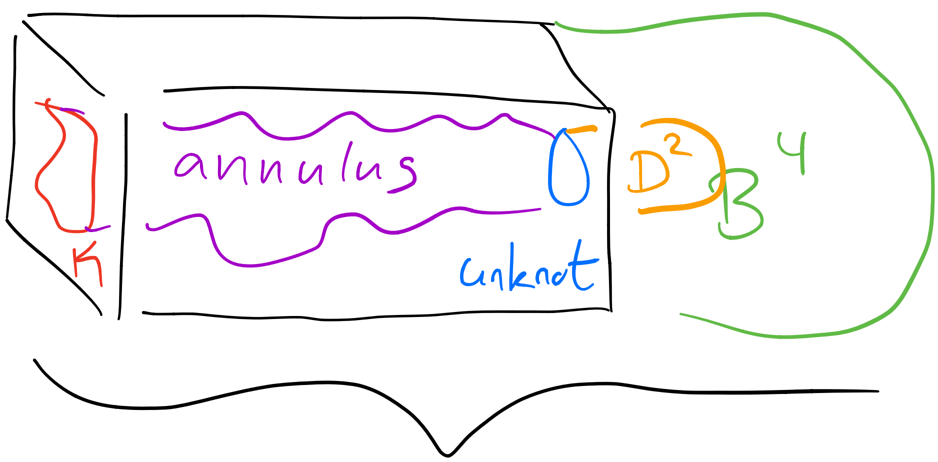
$$Y = \partial(\mathbb{Q}HB^4)$$



Let $\mathcal{C} = \{\text{knots}\}/_{\text{smooth concordance}}$



So $[K] = [\text{unknot}] := 0$ iff K is slice



$B^4 \supset D$

Then get homomorphism

$$\varphi : \mathcal{C} \longrightarrow \prod_{q \in Q} \Theta^3_{\mathbb{Q}}$$

$$[K] \longmapsto \text{list of all } [\Sigma_q(K)]$$

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(Observations 1 + 2 were that $\varphi(0) = 0$, other properties of homomorphism are similar)

Motivating question: to what extent does φ characterize slice knots? Is $\operatorname{Ker} \varphi$ nontrivial?

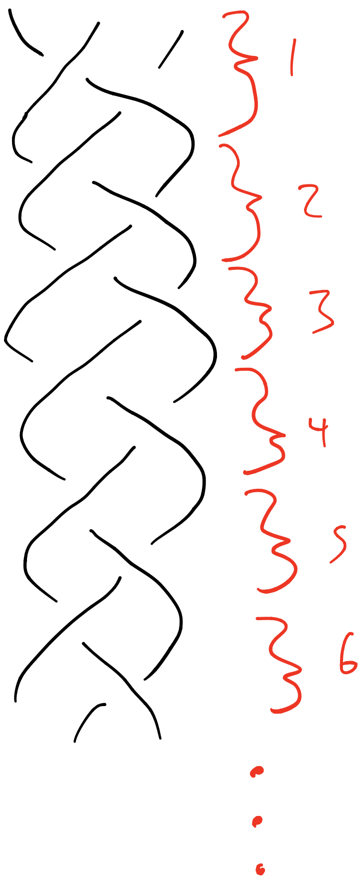
i.e. Do there exist non-slice knots whose Q-fold branched covers bound $\mathbb{Q}HB^4$'s?

(Answer: yes :(sad because this means sliceness is very difficult to obstruct using standard 3-mfd techniques)

Knots for this talk:

$K_n :=$ closure of braid $(\sigma_1 \sigma_2^{-1})^n$

i.e.

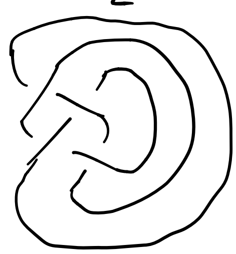


alternating knot with $2n$ crossings

- Also called the $(1,n)$ Turk's head knot
- maybe called a "weave" knot with some indices

(Take n odd and not divisible by 3)

- If n even, then $\Sigma_2(K) \neq \partial \mathbb{Q}HB^4$

  e.g. $K_2 =$  Figure eight

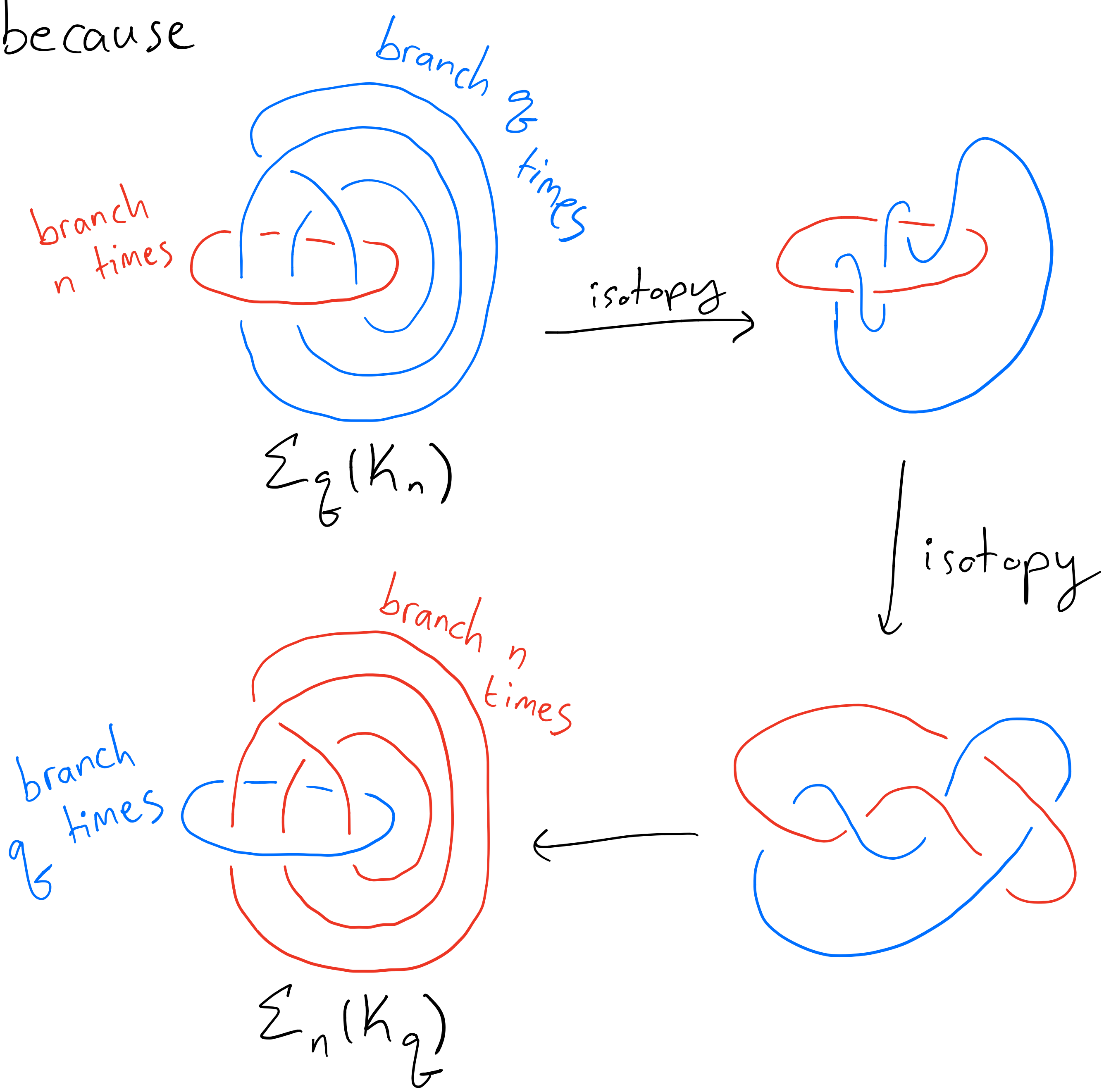
  $|H_1 \Sigma_2| = 5$ not square

- If $3 | n$, then K_n actually a link.

Thm

If $n \neq 2^r, 3^r \in \mathbb{Q}$ and $q \in \mathbb{Q}$, then $\Sigma_q(K_n)$ bounds a $\mathbb{Q}HB^4$.

Pf Note $\Sigma_q(K_n) \cong \Sigma_n(K_q)$ because

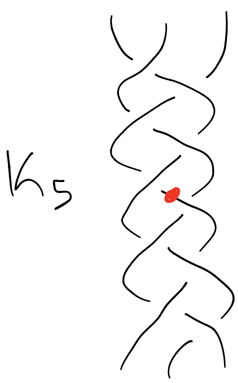


Claim: K_n bounds smooth disk in a $\mathbb{Q}HB^4$ Z with $H_1(Z;\mathbb{Z})$ all 2-torsion

Pf K_n is strongly negative-amphichiral

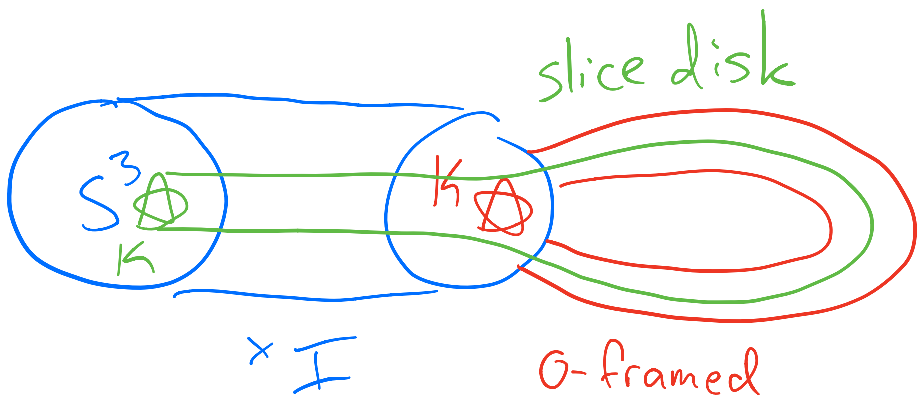
i.e. $\exists$ orientation-reversing involution

$\tau : S^3 \to S^3$ fixing two points of K_n
$K_n \to K_n$



τ = reflection through this point

Can use τ to construct Z



quotient this by ∂ by extension of τ

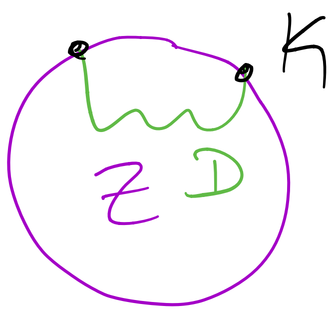
Lemma (Casson Gordon)

If q odd prime power,

then $K_n = \partial(\text{disk into } \mathbb{Z}/_2\, HB^4)$

$\Rightarrow \Sigma_q(K_n) = \partial(\mathbb{Q}\, HB^4)$

Pf Take cover of



Z branched over D

q and orders in

$H_1(Z;\mathbb{Z})$ coprime

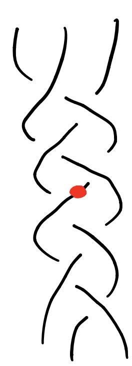
will $\Rightarrow$ cover is a $\mathbb{Q}HB^4$

Back to

<u>Thm</u>
If $n \neq 2^r, 3^r \in \mathbb{Q}$ and $q \in \mathbb{Q}$, then $\Sigma_q(K_n)$ bounds a $\mathbb{Q}HB^4$.

<u>Pf</u>
If q odd, then claim follows from K_n strongly negative-amphichiral + Casson-Gordon

If $q = 2^r$, then $\Sigma_{2^r}(K_n) \cong \Sigma_n(K_{2^r})$ bounds a $\mathbb{Q}HB^4$ since K_{2^r} strongly negative-amphichiral + Casson-Gordon.



So far:

map $\varphi : \mathcal{C} \to \prod_{\mathbb{Q}} \Theta^3_{\mathbb{Q}}$

proved $[K_n] \in \ker \varphi$

if n an odd prime power, $3 \nmid n$.

Thm $K_7, K_{11}, K_{17}, K_{23}$ not slice.

$(\Rightarrow \ker \varphi \neq 1)$

$\mathbb{Z}_2^4 \subset \ker \varphi$

Rmk K_5 is ribbon

Rmk K_7 not slice due to master's thesis of Sartori 2010, different context

Rmk These knots are also linearly independent in $\mathcal{C}$. Think $\left\{ [K_n] \,\middle|\, \begin{smallmatrix} n \equiv 5 \bmod 6 \\ n \geq 11 \end{smallmatrix} \right\}$ are all linearly independent, but can only obstruct finitely many.

Rmk $K_n \# K_n$ slice since $K_n = -\overline{K_n}$.

Slice obstruction: twisted Alexander polynomials

Take $M_K := S^3_0(K)$
ξ_d = d-th root of unity

representation
$\alpha : \pi_1 M_K \to GL(q, \mathbb{Q}[\xi_d][t^{\pm 1}])$
gives twisted Alexander module
$A^\alpha(K) := H_1(M_K; \mathbb{Q}[\xi_d][t^{\pm 1}]^q)$
a $\mathbb{Q}[\xi_d][t^{\pm 1}]$ module

If $q = d = 1$, then $\alpha : \pi_1 M_K \to GL(1, \mathbb{Q}[t^{\pm 1}])$
and $A^\alpha(K)$ is the classical rational Alexander module.

$$A^\alpha(K) = \frac{\mathbb{Q}[\xi_d][t^{\pm 1}]}{(\text{twisted Alexander ideal})}$$

Generator = twisted Alexander polynomial

Write $\tilde{\Delta}^{\alpha}_{K}(t) = \alpha$-twisted Alexander polynomial of K.

- Gross algebraic object
- related to representation theory of $\pi_1(S^3 \setminus K)$
- difficult to compute but $\exists$ some implementation due to
  1) Kirk - Livingston
  2) (Allison N.) Miller - Powell

Extremely useful theorem: Generalized Fox-Milnor

If K slice, then $\tilde{\Delta}^{\alpha}_{K}(t)$ factors as a norm

Specifically, $q \in \mathbb{Q}$, then $|H_1 \Sigma_q(K)| = n^2$ and $\exists\, P \subset H_1(\Sigma_q K)$, $|P| = n$ metabolizer and representation vanishing on P (didn't explain what this means) with $\tilde{\Delta}^{\alpha}_{K}(t) = ct^k f(t) \overline{f(t)}$

Sketch of why K_n $n \in \{7, 11, 17, 23\}$ is not slice:

- Understand metabolizers of $H_1 \Sigma_3(K_n)$ (square-root order subgroups fixed by covering transformation and on which linking form vanishes) (i.e. possible P's)

- Compute corresponding twisted Alexander polynomials

- Obstruct factorization in $\mathbb{Q}[\xi_d][t^{\pm 1}]$ by arguing sufficient to obstruct in $\mathbb{Z}/p[t^{\pm 1}]$ and then use Maple

Get linear independence from factoring

$$H_1(\Sigma_3(K_{n_1} \# \cdots \# K_{n_m})) \cong H_1(\Sigma_3 K_{n_1}) \oplus \cdots \oplus H_1(\Sigma_3 K_{n_m})$$

representation here $\rightsquigarrow$ rep on each summand

Question

What strategy can possibly

1) show $(\mathbb{Z}/2)^{\infty} \subset \operatorname{Ker} \varphi$?

(Unlikely to simultaneously compute infinitely many $\tilde{\Delta}^{\alpha}_{K}(t)$'s)

2) Is $\mathbb{Z} \subset \operatorname{Ker} \varphi$?

(we used strong negative-amphichirality to show $[K_n] \subset \operatorname{Ker} \varphi$, which forced $[K_n \# K_n] = [\text{unknot}]$)