Topology in Dimension 4.5 – Session C
Motivation and Background

Ryan Budney

University of Victoria
rybu@uvic.ca

November 3, 2022

Motivation & Background

Q: Why study diffeomorphism groups?

Motivation & Background

Q: Why study diffeomorphism groups?

A_1: The homotopy-type of diffeomorphism groups are related to some of the most basic features of manifold theory.

eg_1: The homotopy-equivalence $\mathrm{Diff}(S^1 \times D^1) \simeq \mathbb{Z}$ is (largely) a manifestation of the linking number and Schönflies theorem.

In this presentation $\mathrm{Diff}(M)$ denotes all diffeomorphisms of M that restrict to the identity on ∂M.

Motivation & Background

Q: Why study diffeomorphism groups?

A_1: The homotopy-type of diffeomorphism groups are related to some of the most basic features of manifold theory.

eg_2: $\mathrm{Diff}(S^1 \times D^2) \simeq \{*\}$ is a manifestation of Dehn's Lemma and Alexander's Theorem.

Motivation & Background

Q: Why study diffeomorphism groups?

A_1: The homotopy-type of diffeomorphism groups are related to some of the most basic features of manifold theory.

eg_3: $\mathrm{Diff}(S^1 \times D^{n-1})$ acts transitively on $\mathrm{Emb}(D^{n-1}, S^1 \times D^{n-1})$.

$\mathrm{Emb}(D^{n-1}, S^1 \times D^{n-1})$ is the space of smooth embeddings $D^{n-1} \to S^1 \times D^{n-1}$ that restrict to the standard inclusion $(\{1\} \times D^{n-1})$, on the boundary. This result is true **for all** n.

Motivation & Background

Q: Why study diffeomorphism groups?

A_2: Diffeomorphism (families) are used to describe smooth bundles, clutching map constructions, etc.

eg$_1$: $\pi_0 \mathrm{Diff}(D^{n-1})$ is isomorphic to the group of oriented homotopy n-spheres, provided $n \geq 6$.

Motivation & Background

Q: Why study diffeomorphism groups?

A_3: Determining the structure of diffeomorphism groups of manifolds is one of the few remaining big open problems in high-dimensional manifold theory.

eg_1: 'There is no compact manifold M of dimension 4 or larger for which we know the homotopy-type of $\mathrm{Diff}(M)$.' (Allen Hatcher)

Motivation & Background

Q: Why study diffeomorphism groups?

A_3: Determining the structure of diffeomorphism groups of manifolds is one of the few remaining big open problems in high-dimensional manifold theory.

eg_2: 'We choose to go to the moon in this decade and do the other things, not because they are easy, but because they are hard.' (John F. Kennedy)

Motivation & Background (Dim 1)

Theorem: The inclusion

$$O_2 \to \mathrm{Diff}(S^1)$$

is a homotopy-equivalence.

Proof uses the 'straight-line homotopy'.

More geometrically, the 'elastic bending energy' functional (Kusner, J. Sullivan) gives a deformation-retraction of $Maps(S^1, S^1)$ to the 'constant-speed subspace.' This deformation-retraction restricts to a deformation-retraction of $\mathrm{Diff}(S^1)$ to O_2.

The question of whether or not the inclusion $O_{n+1} \to \mathrm{Diff}(S^n)$ is a homotopy-equivalence is often called the **Smale Conjecture** (for spheres).

Motivation & Background (Dim 2)

The main results in dimension two are:

- $\mathrm{Diff}(S^2) \simeq O_3$
- $\mathrm{Diff}(S^1 \times S^1) \simeq S^1 \times S^1 \times GL_2\mathbb{Z}$
- $\mathrm{Diff}(\Sigma_g) \simeq \pi_0 \mathrm{Diff}(\Sigma_g)$ for $g \geq 2$, i.e. $\mathrm{Diff}(\Sigma_g)$ has contractible components.

Motivation & Background (Dim 2)

A comment on the proofs

Earle-Eells (1967) geometric approach uses the fibre-sequence

$$\mathrm{Diff}_0(\Sigma) \to \mathbb{C}(\Sigma) \to T(\Sigma)$$

where $T(\Sigma)$ is the Teichmuller space associated to the surface Σ and $\mathbb{C}(\Sigma)$ is the space of complex structures on Σ.

Smale (1959)-Gramain (1973) cut-and-paste approach, one considers fiber sequences

$$\mathrm{Diff}(\Sigma) \to \mathrm{Emb}(S^1, \Sigma) \qquad \mathrm{Diff}(\Sigma) \to \mathrm{Emb}(I, \Sigma)$$

which reduce the study of $\mathrm{Diff}(\Sigma)$ to embeddings of curves in a surface, and by induction to $\mathrm{Diff}(D^2) \simeq \{*\}$ (Smale).

Motivation & Background (Dim 3)

'Tell me your 3-manifold M and I can tell you the homotopy-type of Diff(M).'

These results have two forms:

Generalized Smale conjectures: $\mathrm{Diff}(M)$ has the homotopy-type of a (usually) compact subgroup of automorphisms, provided M is a geometric manifold. Typically this subgroup is $\mathrm{Isom}(M)$. The top-level results of this form are due to Hatcher, Gabai, Bamler-Kleiner (unpublished), but this builds on the work of many others, including: Waldhausen, Ivanov, Rubinstein, Bonahon, Otal, and many others.

For non-geometric manifolds there are theorems that describe the homotopy-type of $\mathrm{Diff}(M)$ in terms of its geometric decomposition and $\mathrm{Diff}(N_i)$ where N_i are the irreducible or atoroidal bits. In the case of the connect-sum decomposition there is the work of César de Sá, Rourke, Hendriks and Laudenbach, which give non-compact automorphism subgroups in general. In the case of incompressible surfaces there is the work of Hatcher and Ivanov.

Motivation & Background (high dimensions)

The **Cerf-Morlet Comparison Theorem**,

$$\mathrm{Diff}(D^n) \simeq \Omega^{n+1}\left(PL_n/O_n\right).$$

This theorem is mostly used as a device to compare the homotopy groups of PL_n and O_n, i.e. at present we have no direct method of analysing the homotopy of PL_n, the space of PL automorphisms of $\mathbb{R}^n$.

In proper context this should be viewed as a precursor to **smoothing theory**, i.e. this has a more natural interpretation as a homotopy description of the space of smooth structures on D^n.

Motivation & Background (high dimensions)

Definition: A pseudo-isotopy diffeomorphism of a manifold N is a diffeomorphism of $I \times N$ that is the identity on $\{0\} \times N \cup I \times \partial N$.

Such a diffeomorphism would be an isotopy (to the identity map) provided the level-sets $\{t\} \times N$ for $t \in I$ were preserved, explaining the usage of **pseudo**.

$$\mathrm{PDiff}(N) = \{f : \text{pseudoisotopy diffeo of } N\}.$$

There is a fibre-bundle

$$\mathrm{Diff}(I \times N) \to \mathrm{PDiff}(N) \to \mathrm{Diff}(N)$$

called the **pseudo-isotopy fiber sequence**.

Motivation & Background (high dimensions)

Theorem: (Hatcher-Wagoner) assuming $n \geq 6$,

$$\pi_0 \mathrm{Diff}(S^1 \times D^{n-1}) \simeq \pi_0 \mathrm{Diff}(D^n) \oplus \pi_0 \mathrm{Diff}(D^{n-1}) \oplus \bigoplus_{\infty} \mathbb{Z}_2.$$

The infinite-rank 2-torsion factor on the right is the image of the pseudo-isotopy fiber sequence

$$\pi_0 \mathrm{Diff}(I \times S^2 \times D^{n-1}) \longrightarrow \pi_0 \mathrm{PDiff}(S^1 \times D^{n-1}) \xrightarrow{!} \pi_0 \mathrm{Diff}(S^1 \times D^{n-1}) .$$

Motivation & Background (high dimensions)

Theorem: (Hatcher-Wagoner) assuming $n \geq 6$,

$$\pi_0 \mathrm{Diff}(S^1 \times D^{n-1}) \simeq \pi_0 \mathrm{Diff}(D^n) \oplus \pi_0 \mathrm{Diff}(D^{n-1}) \oplus \bigoplus_{\infty} \mathbb{Z}_2.$$

There is a homotopy-equivalence

$$\mathrm{Diff}(S^1 \times D^{n-1}) \simeq \mathrm{Diff}(D^n) \times \mathrm{Emb}(D^{n-1}, S^1 \times D^{n-1}).$$

Hatcher-Wagoner is further saying that

$$\pi_0 \mathrm{Emb}(D^{n-1}, S^1 \times D^{n-1}) \simeq \pi_0 \mathrm{Diff}(D^{n-1}) \oplus \bigoplus_{\infty} \mathbb{Z}_2.$$

Motivation & Background (high dimensions)

Theorem: (Cerf) $\mathrm{PDiff}(M)$ is connected provided $m \geq 5$ and M is simply-connected.

Corollary: Every diffeomorphism of a simply-connected manifold M of dimension $m \geq 6$ that has an interval factor ($M \simeq N \times I$) is isotopic to one that is level-preserving in the I-factor.

Motivation & Background (high dimensions)

One of Cerf's central constructions is the observation that one can almost reconstruct elements of $f \in \mathrm{PDiff}(M)$ from the composite:

$$\begin{array}{ccc} I \times M & \xrightarrow{f} & I \times M \\ & \searrow^{\pi \circ f} & \downarrow \pi \\ & & I \end{array}$$

This gives a homotopy-equivalence between $\mathrm{PDiff}(M)$ and the space of smooth functions $I \times M \to I$ without critical points (with the appropriate boundary conditions), allowing us to think of $\mathrm{PDiff}(M)$ as the non-singular strata of the space of smooth functions $I \times M \to I$.

Motivation & Background (high dimensions)

Theorem: (Hatcher, Igusa) The inclusion map $\mathrm{PDiff}(M) \to \mathrm{PDiff}(M \times I)$ induces an isomorphism of homotopy groups in the range $i < \min\{\frac{m-4}{3}, \frac{m-7}{2}\}$.

Now this is known as the **Igusa Stable Range**. It was stated incorrectly in Hatcher's Higher Simple Homotopy Theory and later proven in Igusa's Ph.D thesis.

Motivation & Background (high dimensions)

Definition: The collection of diffeomorphisms of $\Delta^k \times M$ that restrict to the identity on $\Delta^k \times \partial M$ and preserve the faces $d_i\Delta^k \times M$, as a simplicial set, is called the space of **block diffeomorphisms** of M, $\widetilde{\mathrm{Diff}}(M)$.

Theorem: (Hatcher) Spectral sequence for computing the homotopy groups of $\widetilde{\mathrm{Diff}}(M)/\mathrm{Diff}(M)$ in terms of the homotopy of $\mathrm{PDiff}(M \times D^k)$.

Motivation & Background (high dimensions)

Theorem: (Hatcher*, Igusa) There is a map

$$\mathrm{PDiff}(M) \to \Omega Wh(M)$$

that is an isomorphism on homotopy groups in the Igusa stable range.
The space $Wh(M)$ is the Whitehead space of M, sometimes called 'higher simple homotopy theory'. $Wh(M)$ is the classifying space of the 'category of spaces with arrows the simple homotopy equivalences' interpreted appropriately.

* Hatcher stated this in PL category, but proof was wrong. Igusa stated and proved in smooth category. Burghelea later gave a PL version of theorem for smoothable PL-manifolds.

Theorem: (Casson, Sullivan, Wall, Quinn, Ranicki) The homotopy of

$$HomEq(M)/\widetilde{\mathrm{Diff}}(M)$$

is computable via L-theory (a spectrum).

i.e. The gap between $\mathrm{Diff}(M)$ and $\widetilde{\mathrm{Diff}}(M)$ controlled by pseudo-isotopy.
The gap between $\widetilde{\mathrm{Diff}}(M)$ and $HomEq(M)$ controlled by L-theory.

Theorem: (Quinn) Homotopic diffeomorphisms of a closed simply-connected smooth 4-manifold are isotopic after taking a connect-sum with (perhaps several copies of) $S^2 \times S^2$.

Theorem: (Ruberman) Stabilization can be necessary.