Three views of the Freedman–Quinn invariant

Rob Schneiderman

Lehman College CUNY & MPIM

BIRS Topology in Dimension 4.5
November 2022

Based on joint work with Peter Teichner:
'Homotopy versus Isotopy: spheres with duals in 4-manifolds' – sections 4 and 7 – arXiv:1904.12350v4 [math.GT]

This talk will describe a version of the Freedman–Quinn invariant $\mathrm{fq}(R, R')$ for homotopic embedded 2-spheres R, R' in a 4-manifold M.

$\mathrm{fq}(R, R')$ takes values in a quotient of $\mathbb{F}_2 T$, the $\mathbb{Z}/2\mathbb{Z}$-vector space on the 2-torsion elements $T := \{g \in \pi_1 M \mid g^2 = 1 \neq g\} < \pi_1 M$.

$\mathrm{fq}(R, R')$ vanishes if R and R' are concordant.

Origin of fq

A more general version of fq originally appeared in the setting of the classification of simply connected 4-manifolds in the proof of Theorem 10.9 of Freedman and Quinn's book *Topology of 4-manifolds*.

The "fq" notation, along with a generalization/correction, is in R. Stong's paper *Uniqueness of* π_1*-negligible embeddings in 4-manifolds: A correction to Theorem 10.5 of Freedman and Quinn*.

Michael Klug's talk will present Stong's generalization.

This talk will sketch three views of fq

6-dimensional view $\rightsquigarrow$ easy proof of well-definedness

5-dimensional view $\rightsquigarrow$ Stong's generalization

4-dimensional view $\rightsquigarrow$ 4d smooth Light Bulb Theorem

For simply connected A^3, B^3 properly immersed in X^6,
have *intersection invariant*

$$\lambda(A,B) = \sum_{p \in A \pitchfork B} \epsilon_p \cdot g_p \in \mathbb{Z}\pi_1 X$$

and *self-intersection* invariant

$$\mu(A) := [\sum_{p \in A \pitchfork A} \epsilon_p \cdot g_p] \in \mathbb{Z}\pi_1 X / \langle g + g^{-1}, 1 \rangle$$

with $\epsilon_p = \pm$ the usual sign at p,
and $g_p \in \pi_1 X$ determined by choosing loop that changes sheets at p.
(Smooth, compact, oriented, based,...)

Intersection form in 6-dimensions

λ and μ are invariant under homotopy (rel boundary).

$\mu(A) = 0$ if A is homotopic to an embedding (rel boundary).

Intersection form in 6-dimensions

Have relations:

$$\mu(A+B) - \mu(A) - \mu(B) = [\lambda(A,B)]$$

and

$$\lambda(A,A) = \mu(A) - \overline{\mu(A)} \in \mathbb{Z}\pi_1 M/\langle 1\rangle$$

where $\bar{g} := g^{-1}$ on $\mathbb{Z}\pi_1 X$.

The Freedman–Quinn invariant fq of a homotopy – Definition

Let H be a homotopy between embedded 2-spheres R and R' in M^4.

Define $\mathrm{fq}(H)$ to be the self-intersection invariant of the 'thickened' track $\widehat{H} : S^2 \times I \looparrowright M \times \mathbb{R} \times I$:

$$\mathrm{fq}(H) := \mu(\widehat{H}) = \sum_{p \in \widehat{H} \pitchfork \widehat{H}} \epsilon_p \cdot g_p \in \mathbb{Z}\pi_1 M / \langle g + g^{-1}, 1 \rangle$$

The Freedman–Quinn invariant fq of a homotopy – Target

Product structure $M \times \mathbb{R} \times I \Rightarrow \lambda(A, B) \equiv 0$.

$0 = \lambda(A, A) = \mu(A) - \overline{\mu(A)} \in \mathbb{Z}\pi_1 M/\langle 1 \rangle \Rightarrow \mathrm{im}(\mu) < \mathbb{F}_2 T$.

So have

$$\mathrm{fq}(H) := \mu(\widehat{H}) \in \mathbb{F}_2 T$$

Recall: $T := \{g \in \pi_1 M \mid g^2 = 1 \neq g\}$.

The Freedman–Quinn invariant $\mathrm{fq}(R, R')$.

For H a based homotopy between embedded 2-spheres $R, R' \subset M^4$, define:

$$\mathrm{fq}(R, R') := [\mathrm{fq}(H)] \in \mathbb{F}_2 T / \mu(\pi_3 M)$$

$\mu : \pi_3 M \cong \pi_3(M \times \mathbb{R} \times I) \to \mathbb{F}_2 T$ is a *homomorphism* by the relation $\mu(A + B) - \mu(A) - \mu(B) = [\lambda(A, B)] = 0$.

To show independence of choice of H, suffices to show that any self-homotopy J of R has $\mu(J) \in \mu(\pi_3 M)$.

This is true because such J agrees with the product self-isotopy on the 2-skeleton of $S^2 \times I$, and the difference is carried by a map of a 3–sphere: $S^3 \looparrowright M \times \mathbb{R} \times I$. (Uses that H is based.)

Defining $\mathrm{fq}(R, R')$ using unbased homotopies?

Question:

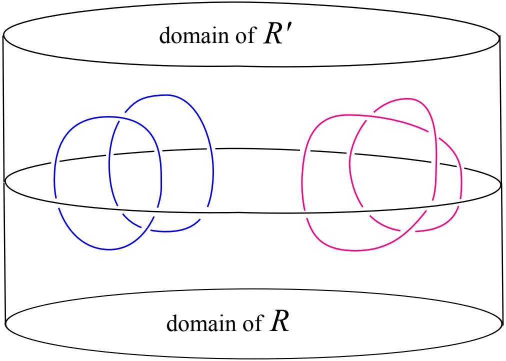
Does there exist a self-homotopy J of some $S^2 \subset M^4$
such that $\mu(J) \notin \mu(\pi_3 M)$?

Answer is "No" if $[S^2] \in \pi_2 M$ has trivial stabilizer in $\pi_1 M$
(eg. if $[S^2]$ admits an algebraic dual).

Answer is "Yes" to analogous question for $S^2 \subset N^5$ and $S^1 \subset Y^3$.

5-dimensional view of fq

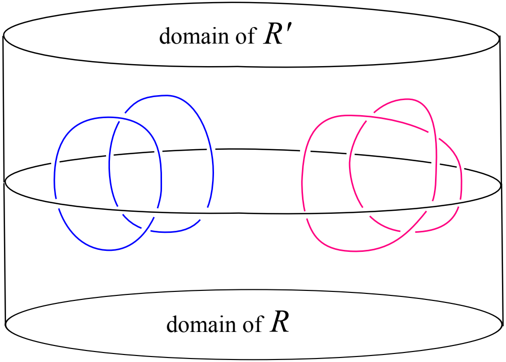
Preimage of each singular circle of the track $S^2 \times I \looparrowright M \times I$ of a regular homotopy from R to R' is a pair of circles or a single circle.



Each circle in a pair maps by homeomorphism onto its image.
Each single circle is a double cover of its image.

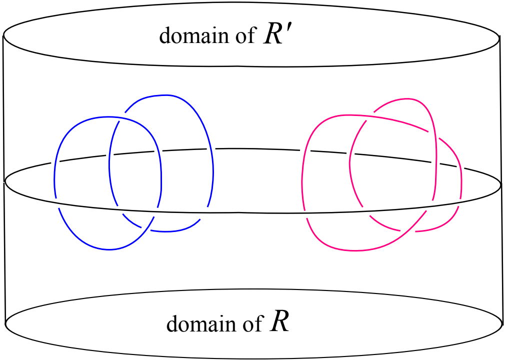
5-dimensional view of fq

Singular circles $\rightsquigarrow$ elements of $\pi_1 M$ from sheet-changing loops.



Singular circles with single circle preimages $\rightsquigarrow g \in \pi_1 M,\ g^2 = 1$.

Define: $\mathrm{fq}(H) := \sum c_i t_i \in \mathbb{F}_2 T$ where the coefficient c_i of t_i is the number modulo 2 of single circles corresponding to $t_i \in T \subset \pi_1 M$.

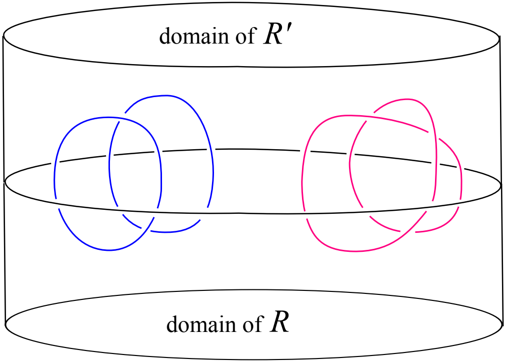


5-dimensional view of fq $\rightsquigarrow$ Stong's invariant

Michael Klug's talk will introduce Stong's secondary invariant, which is extracted from all the singular circles of $S^2 \times I \looparrowright M \times I$ in the case that $\mathrm{fq}(R, R')$ vanishes and R is *spherically characteristic*.

From 5-d view back to 6-d view

Can eliminate each pair of circles after the thickening
$S^2 \times I \looparrowright M \times I \quad \mapsto \quad S^2 \times I \looparrowright M \times \mathbb{R} \times I$
by pushing one sheet into the $\mathbb{R}$-direction.



Pushing in the $\mathbb{R}$-direction: single circle $\mapsto$ single self-intersection.

Setting up 4-d view of fq

A regular homotopy from $R \subset M$ to $R' \subset M$ can be described, up to isotopy, as

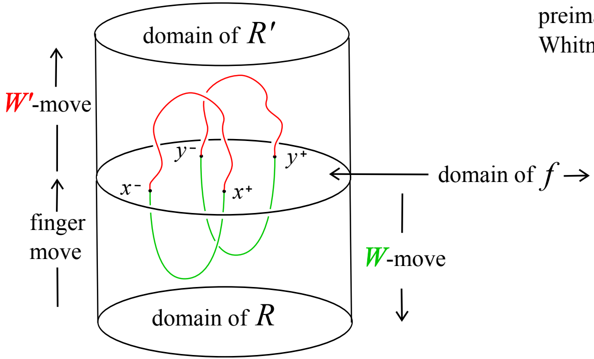
finger moves on R, yielding an immersed 'middle level' $f : S^2 \looparrowright M$,

followed by Whitney moves on f yielding R'.

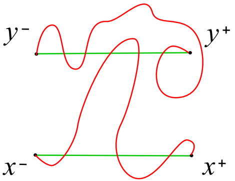
The Whitney moves leading from f to R'
are guided by ascending Whitney disks on f.

Whitney moves leading back from f to R
(which are 'inverse' to the finger moves)
are guided by descending Whitney disks on f.

From 5-d view to 4-d view of fq (with new color scheme)

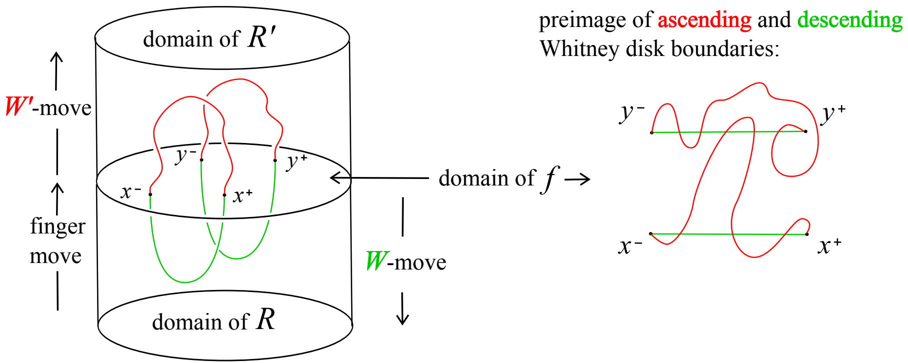


preimage of ascending and descending Whitney disk boundaries:



$$\text{in } M \times \{*\} : \quad f(x^+) = f(y^+) \quad \text{and} \quad f(x^-) = f(y^-)$$

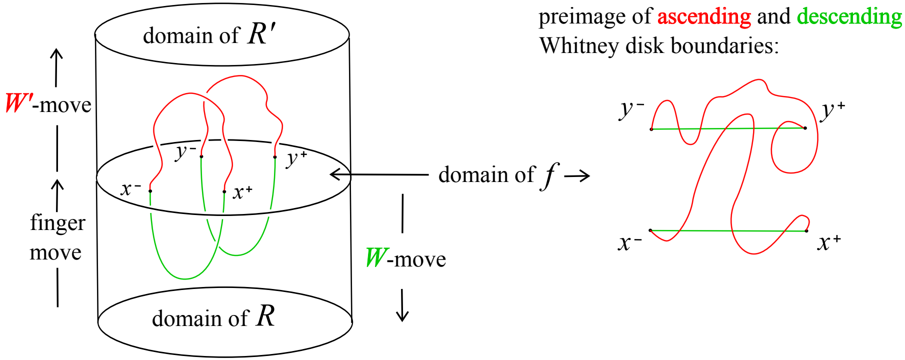
From 5-d view to 4-d view of fq (with new color scheme)



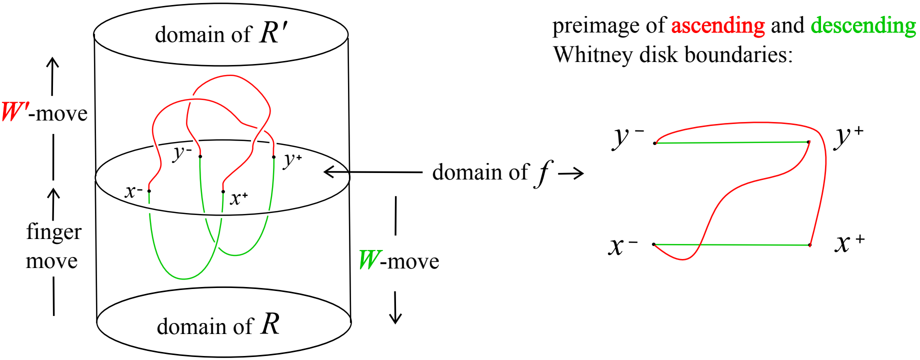
The <u>singular circles</u> of the track $S^2 \times I \looparrowright M \times I$ of a regular homotopy from R to R' ‘project’ to <u>boundaries</u> of ascending Whitney disks W_i' (leading to R') and descending Whitney disks W_i (inverse to finger-moves on R) in a ‘middle level’ $f : S^2 \times \{*\} \looparrowright M \times \{*\}$.

In middle level $f : S^2 \times \{*\} \looparrowright M \times \{*\}$ of $S^2 \times I \looparrowright M \times I$

Preimage circle **pairs** project to **pairs** of immersed circles formed by preimages of ascending and descending Whitney disk boundaries:

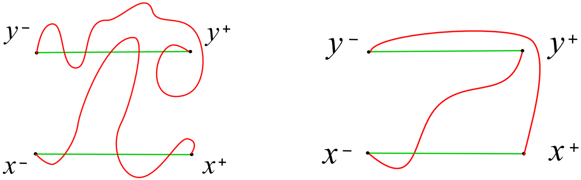


Preimage **single** circles project to **single** immersed circles formed by preimages of ascending and descending Whitney disk boundaries:



Define: $\mathrm{fq}(H) := \sum c_i t_i \in \mathbb{F}_2 T$ where the coefficient c_i of t_i is the number modulo 2 of immersed circles formed by preimages of ascending and descending Whitney disk boundaries corresponding to t_i in a middle level $f : S^2 \times \{*\} \looparrowright M \times \{*\}$ of H.

preimage of ascending and descending Whitney disk boundaries in domain of f:



Theorem: [Gabai, S.–Teichner] Homotopic 2-spheres $R, R' \subset M^4$ admitting a common geometric dual sphere (framed, embedded) are isotopic if and only if $\mathrm{fq}(R, R')$ vanishes.

Proofs use dual to 'clean up' the Whitney disks (including their boundary arcs!) in a middle level of a homotopy...

Dave Gabai will explain how the *Dax invariant* gives a further obstruction to isotopy in the setting of homotopic disks rather than spheres.