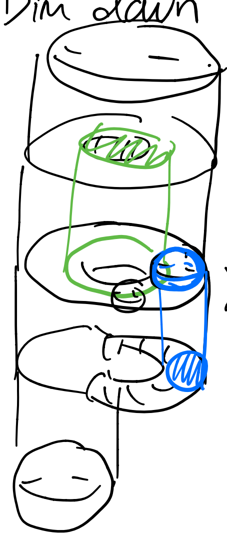
Surfaces in 4-mfds via banded unlink diagrams

- closed Surface $\Sigma \hookrightarrow X^4$ closed smooth

- analogue to Knot theory
  $K \hookrightarrow M^3$
  ~ What can $\pi_1(X^4 \setminus \Sigma)$ be?
  ~ What 4-mfds arise from surgery?

$\{\text{Spheres}\} \hookrightarrow X^4$
$\hookrightarrow$ cobordisms of 4-mfds

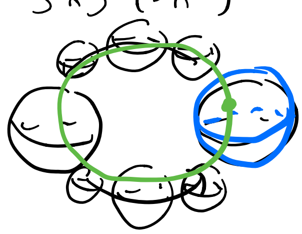
Dim down



belt of 2-handle $= S^2$
attaching of 3-handle $= S^2$

Cobordism $\rightsquigarrow$ described by

$S^1 \times S^2$ $(= X^3)$

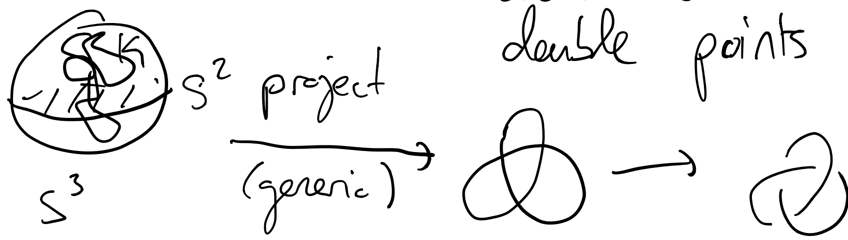


Cobordism of 2-mfds

belt of 1-handle $= S^1$
attaching of 2-handle $= S^1$

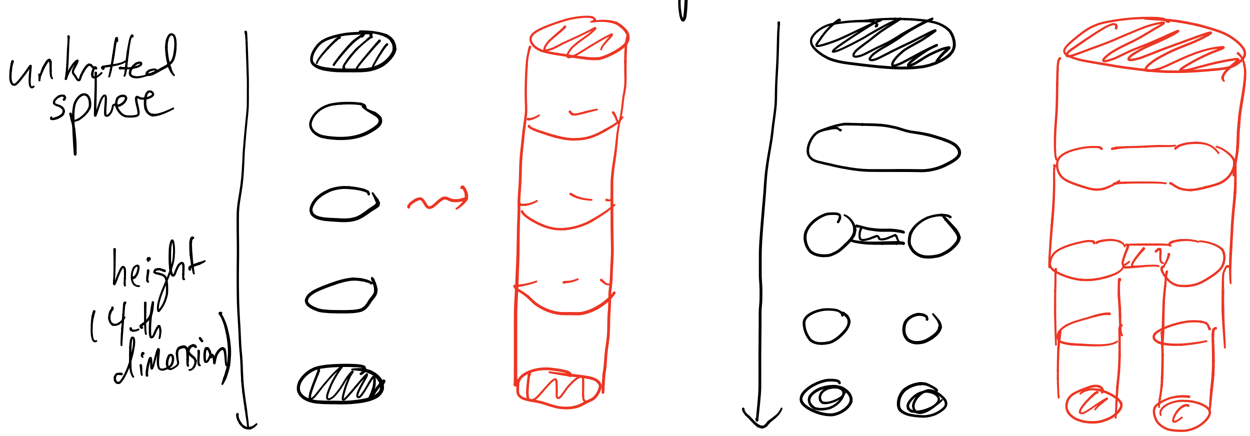
How to describe?

Knot $K \subset S^3 \rightsquigarrow$ project to S^2, break one strand near double points (lower strand)

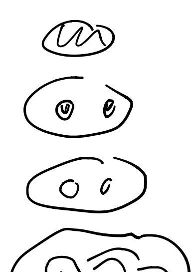


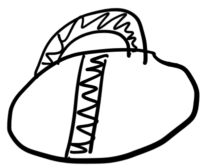
Surface $\Sigma \subset S^4$

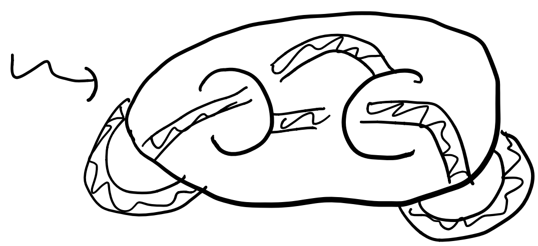
Fox: Movie diagrams



Determined by ∂(minima disks) + bands ($\leftrightarrow$ ind-1 crit pts)





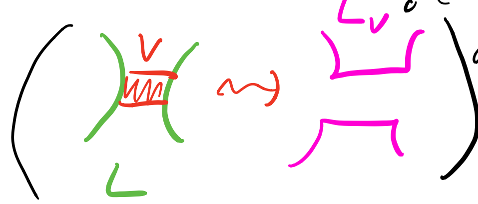


3 min
4 saddles
3 max $\qquad \chi = 3-4+3 = 2$

Def. (L, v) = banded unlink diagram in S^4 if

$h_0 : S^4 \to [0,4]$
$h_0^{-1}(0) \cong h_0^{-1}(4) = pt$
$h_0^{-1}(t) \cong S^3$ $\quad 0<t<4$

min $\quad L$ = unlink in $S^3 \left(= h^{-1}\left(\tfrac{3}{2}\right)\right)$
bands $\quad v$ = bands attached to L
max $\quad L_v$ = unlink in S^3



$(L,v) \overset{\text{determines}}{\rightsquigarrow}$ Surface $\Sigma(L,v)$

Disks bounded by L (pushed into $h^{-1}[0,\frac{3}{2}]$)

$= \cup \underbrace{V}_{\text{bands}} \subset h^{-1}(\frac{3}{2})$

$\cup$ Disks bounded by L_v (pushed into $h^{-1}[\frac{3}{2},4]$)

Fox movie pictures

$\rightsquigarrow$ (Kawauchi Suzuki Shibuya) $\forall \Sigma, \exists (L,v)$ so

$\Sigma \underset{\sim}{\overset{\text{isotopic}}{}} \Sigma(L,v)$

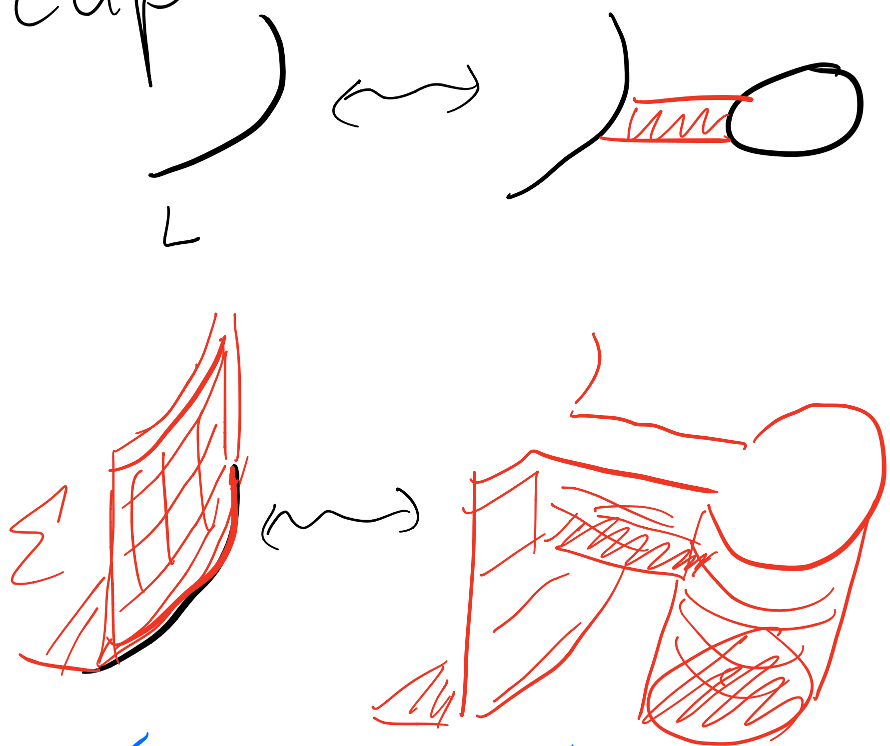
(Say (L,v) a diagram for Σ)

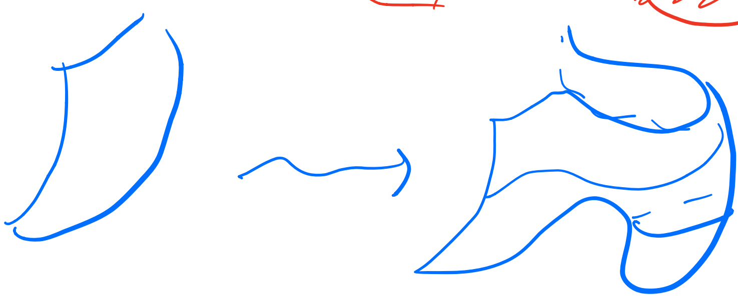
Thm (Conj by Yoshikawa
Proved by Swenton;
Kearton-Kurlin)

If (L_1, v_1) and (L_2, v_2) are diagrams for Σ, then (L_1, v_1) related to (L_2, v_2) by a seq. of

- cup/cap
- band swim
- band slide
- isotopy

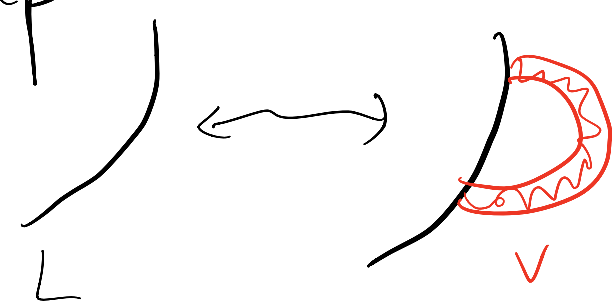
cup

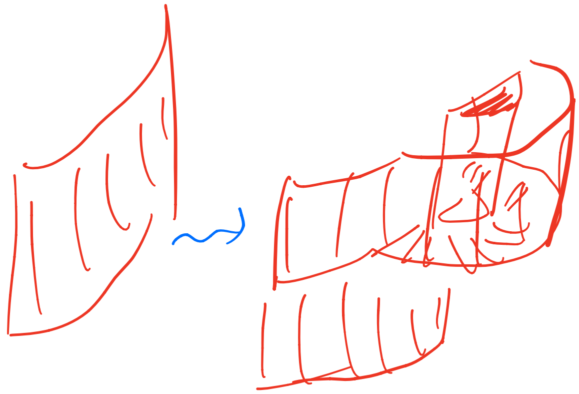


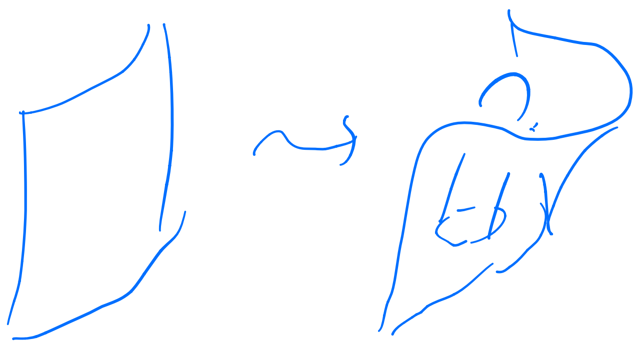


0,1-stabilization of $h|_{\Sigma}$

cap

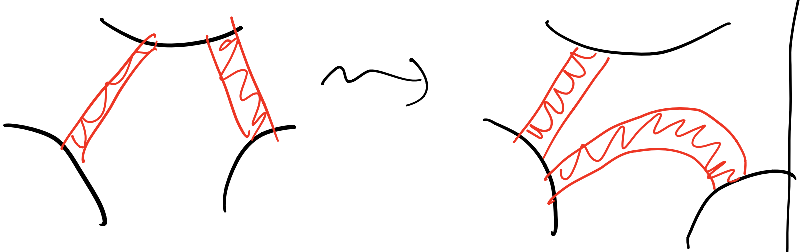






1, 2-stabilization of $h|_\Sigma$

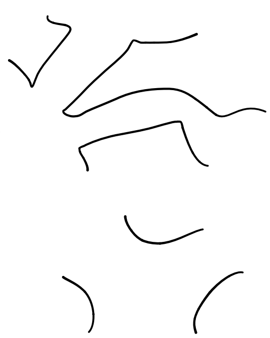
band slide



above

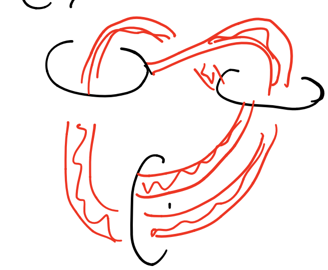
movie

isotopic $\rightsquigarrow$

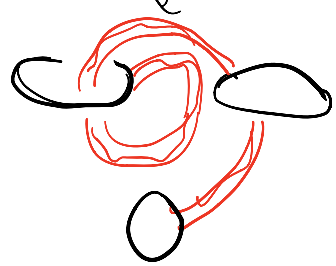


below bands

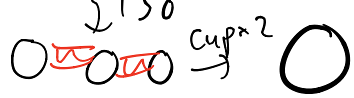
Ex



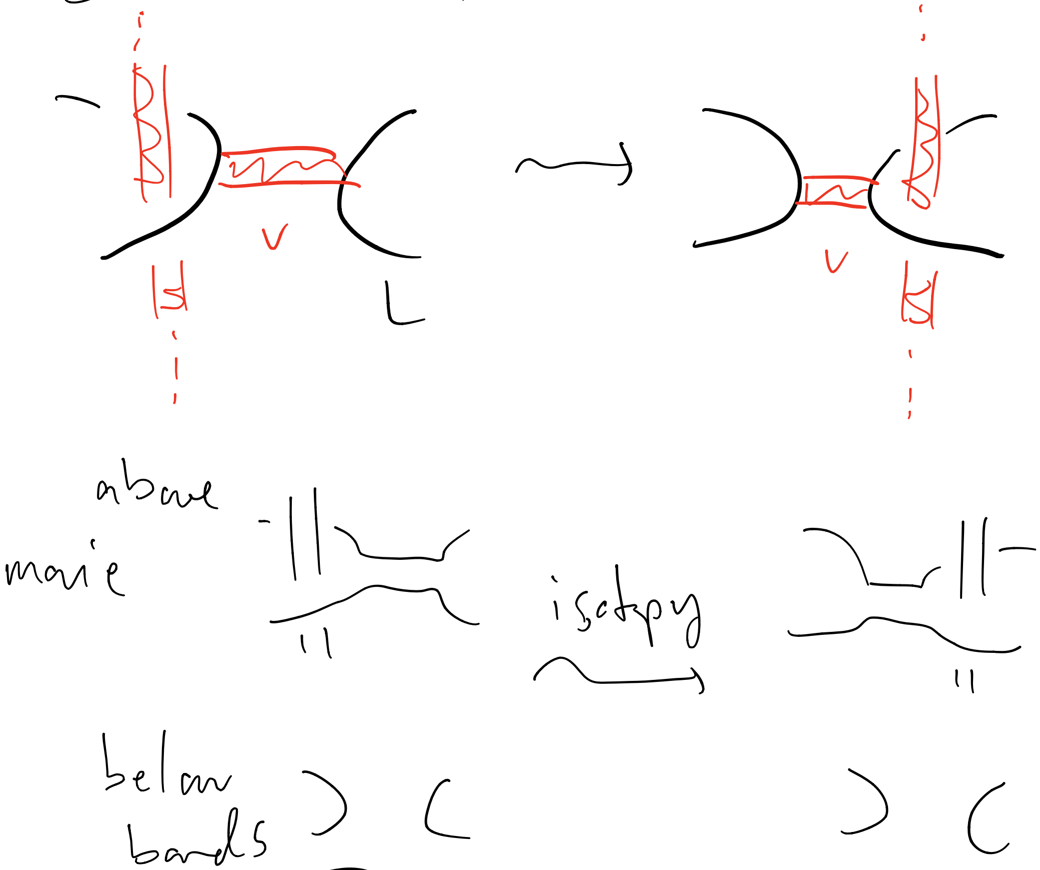
} swim



{ iso



band swim



Extend to general 4-mfd

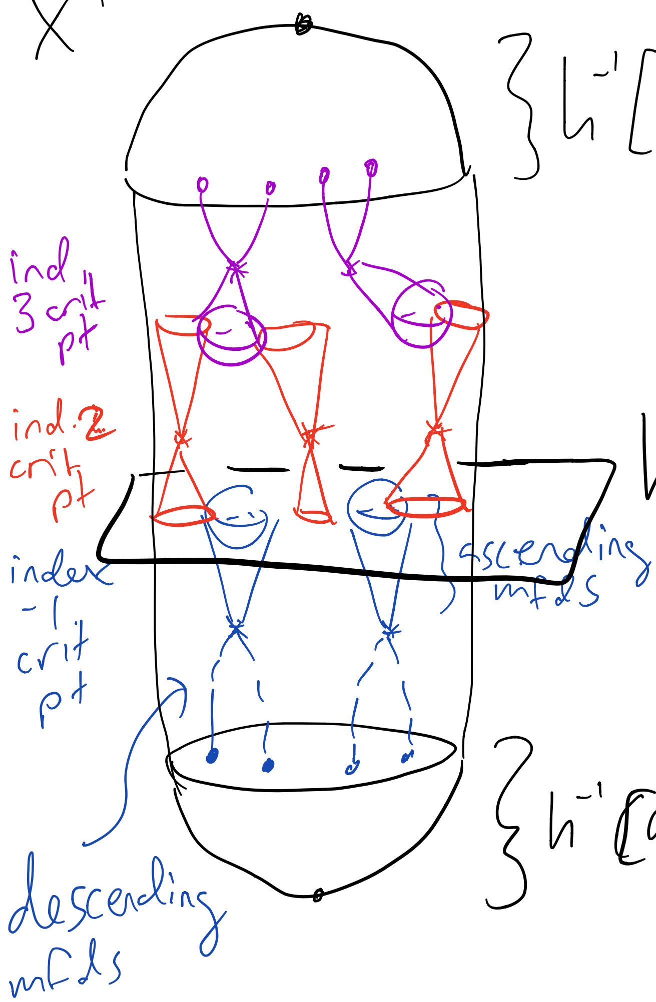
$h : X^4 \to \mathbb{R}$ self-indexing Morse function

(index-i crit pts in $h^{-1}(i)$)

say 1 ind-0
1 ind-4 pt

Draw $\left.\begin{matrix}\text{Kirby}\\ \text{diagram}\end{matrix}\right\}$ of X^4 using h / K

X^4



$\}\ h^{-1}[\tfrac{7}{2}, 4] \cong B^4$

$h^{-1}(\tfrac{3}{2}) \cong \# S^1 \times S^2$

$\supset$ framed circles (intersections w/ desc. mfds of ind-2 crit pts)

surger $S^0 \times B^3$ get $I \times S^2$

$\}\ h^{-1}[0, \tfrac{1}{2}] \cong B^4$

Def Kirby diagram is

$L_1 \sqcup L_2 \subset S^3$ $\rightsquigarrow$ L_1 unlink

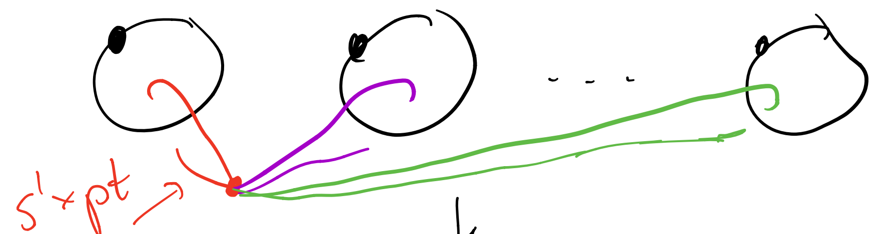
links

L_2 each component has integer framing

so that $S^3_0(L_1 \cup L_2) \cong \#_k S^1 \times S^2$ for some k

To draw $\#_k S^1 \times S^2$

draw k "dotted circles"



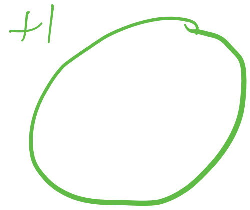
($\#_k S^1 \times S^2 = S^3$ surgered along the k dotted circles ; 0-surgery)

So X^4 determined by

disjoint

dotted circles

+ framed circles $\subset S^3$

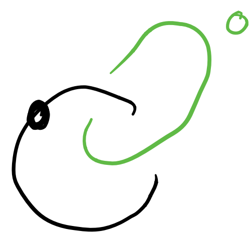
$\mathbb{CP}^2 \rightsquigarrow$ 

1 4-cell
0 3-cells
1 2-cell
no 1-cells
1 0-cell

4-cell

$S^4 \rightsquigarrow$

1 4-cell
no 3-cell
no 2-cell
no 1-cell
1 0-cell

$S^4 \rightsquigarrow$ 

1 4-cell
0 3-cells
1 2-cell
1 1-cell
1 0-cell

Reference (Gompf - Stipsicz)

Now $\Sigma \subset X^4$ $\left(\underset{\text{induces } K}{h : X^4 \to I} \right)$

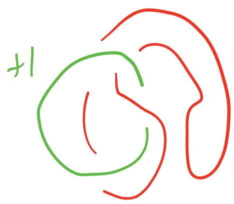
Isotope Σ so min of $h|_\Sigma$ in $h^{-1}[0, \frac{3}{2}]$, saddles of $h|_\Sigma$ in $h^{-1}(\frac{3}{2})$, max of $h|_\Sigma$ in $h^{-1}[\frac{3}{2}, 4]$

Ex)

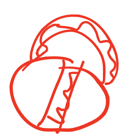
$$\begin{array}{c} \Sigma \subset \mathbb{CP}^2 \\ \cong \\ T^2 \end{array}$$

$\mathbb{CP}^2$

$h = \frac{7}{2}$  } disk (max)

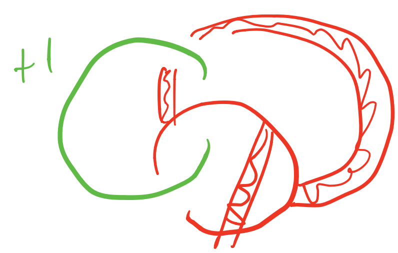
$h = \frac{5}{2}$  $S^3_{\text{unknot}}(1) \overset{\text{surgery}}{\cong} S^3$

$h = \frac{7}{4}$ 

$h = \frac{3}{2}$  S^3

just draw K, $\partial \min h|_\varepsilon$, bands

$h = \varepsilon$  S^3



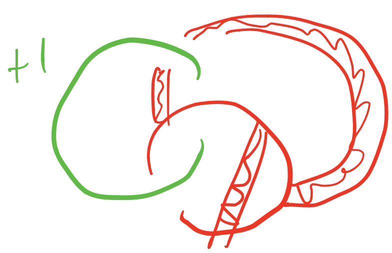
Def. Banded unlink diagram

(K, L, v)

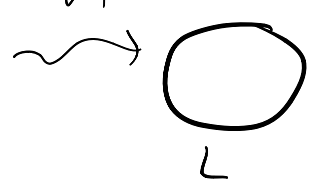
K = Kirby diagram for X^4 induced by h

disjoint from Kirby circles $\begin{cases} L = \text{link} = \text{unlink in } h^{-1}(\frac{3}{2}) \\ v = \text{bands attached to } L \end{cases}$

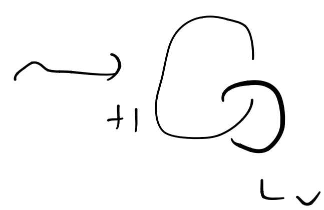
L_v = unlink in $h^{-1}(\frac{5}{2})$



$L \subset h^{-1}(\frac{3}{2})$



$L_v \subset h^{-1}(\frac{5}{2})$



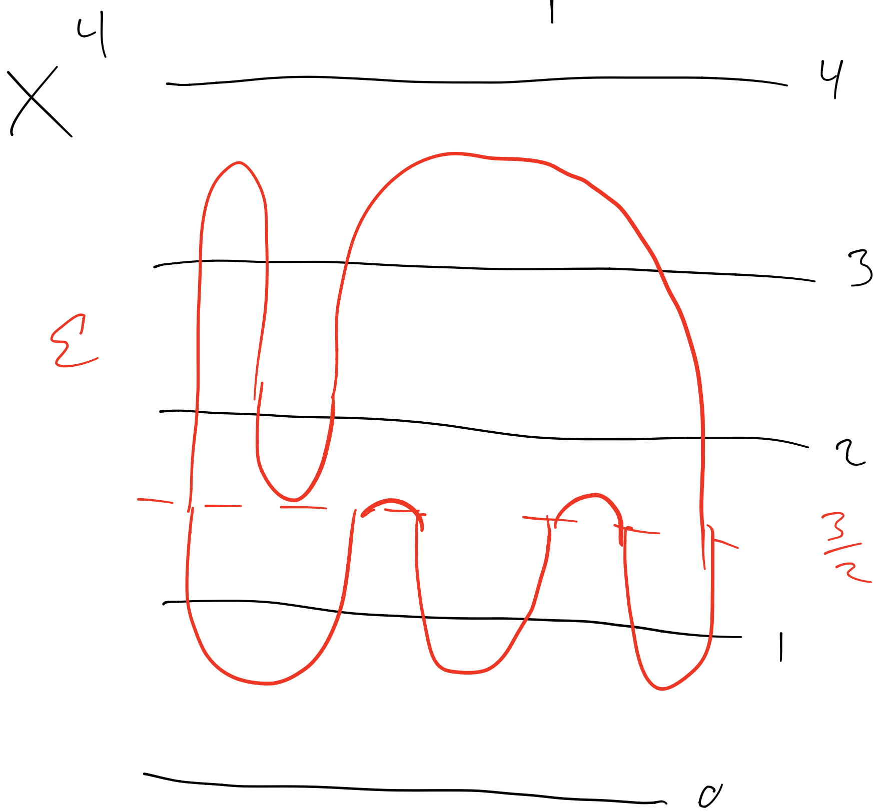
(K, L, v) induces surface

$\Sigma(K, L, v) \subset X^4$

Σ = disks bdd. by L pushed into $h^{-1}[0, \frac{3}{2}]$

$\cup$ bands v in $h^{-1}(\frac{3}{2})$

$\cup$ disks bdd by L_v pushed into $h^{-1}[\frac{3}{2}, 4)$



Say (K, L, v) diagram for Σ if $\Sigma \overset{iso}{\simeq} \Sigma(K, L, v)$

Thm (Hughes-Kim-M)

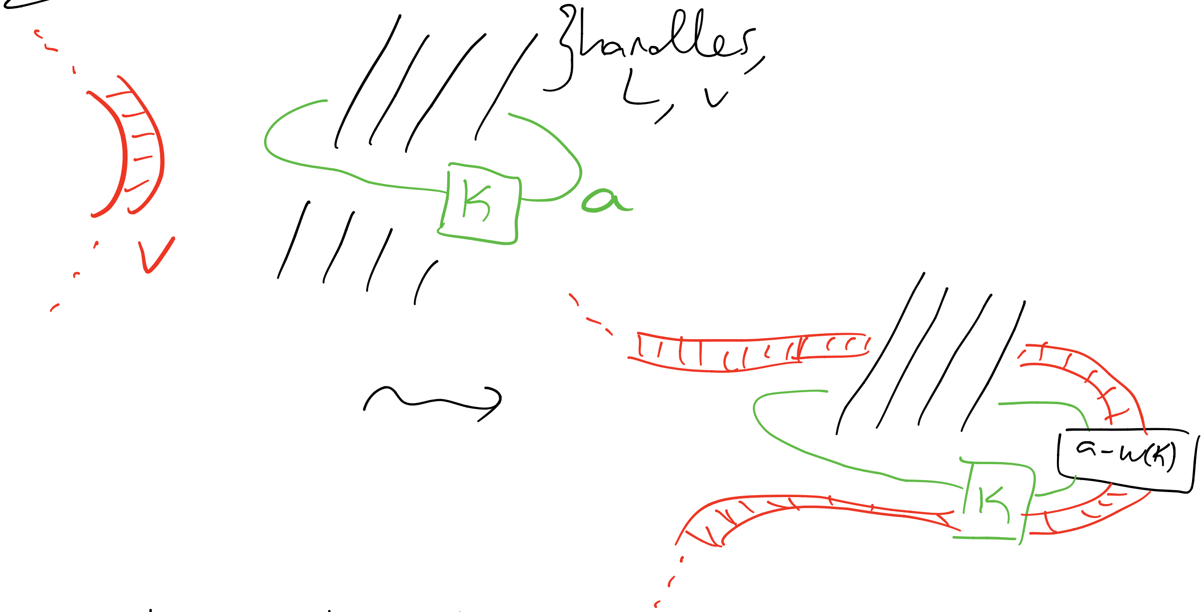
$h: X^4 \to I$ Morse inducing K

$\Sigma \subseteq X^4$ smooth surface

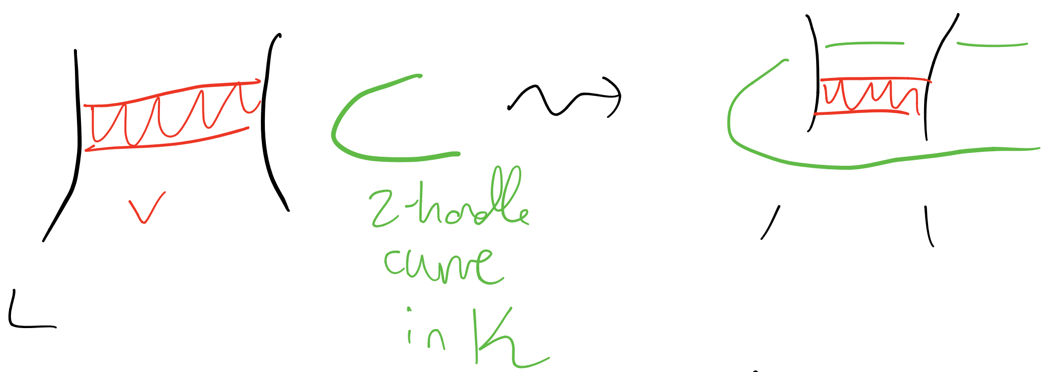
- Σ has a diagram (K, L, v)
- Any two diagrams (K, L_1, v_1) (K, L_2, v_2) for Σ are related by a seq of
  - cup/cap
  - band slide
  - band swim
  - isotopy

  $\Big\}$ "old" moves
  - 2-handle band slide
  - 2-handle band swim
  - dotted circle slides

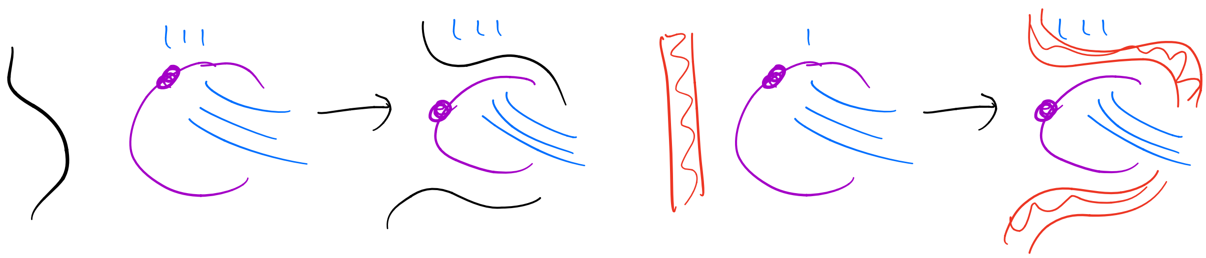
2-handle band slide



2-handle band swim



dotted circle slide



Pf Assume $h|_\Sigma$ Morse

Σ 2-dimensional $\rightsquigarrow$ isotope

$\Sigma \cap h^{-1}[0,\frac{1}{2}]$ up and $\Sigma \cap h^{-1}[\frac{5}{2},4]$ down (so $\Sigma \subset h^{-1}(\frac{1}{2},\frac{5}{2})$)

Drag nbhd of minima of $h|_\Sigma$ down to $h^{-1}(\frac{1}{2})$

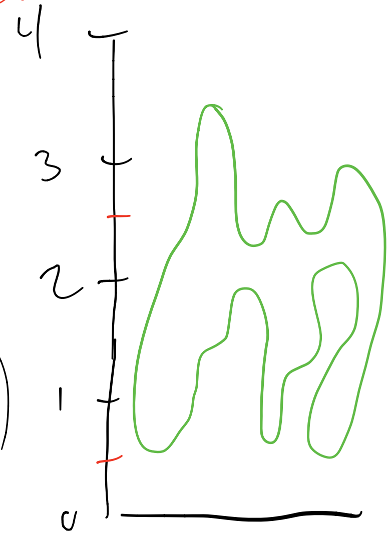
0-dim

(below 1-handles)

nbhd of max of $h|_\Sigma$ up to $h^{-1}([\frac{5}{2},\frac{7}{2}])$

2-dim

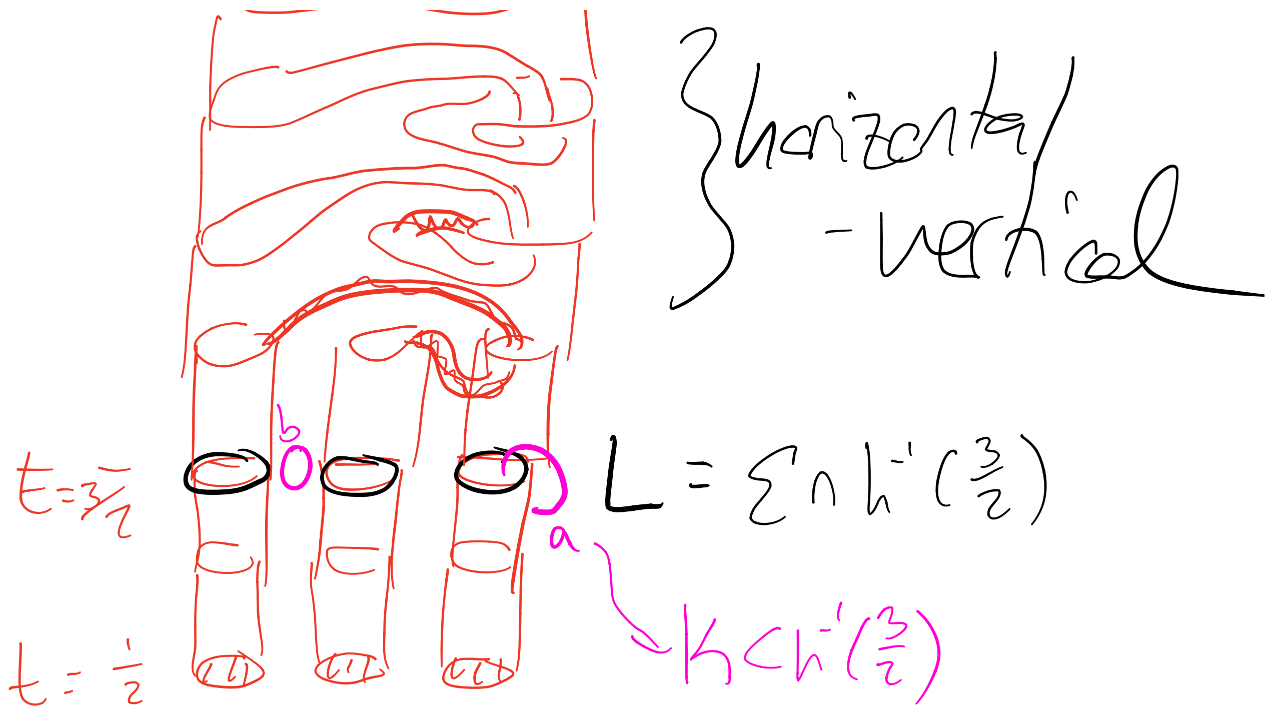
(above 2-handles)



flatten index-1 crit + take to be at distinct heights

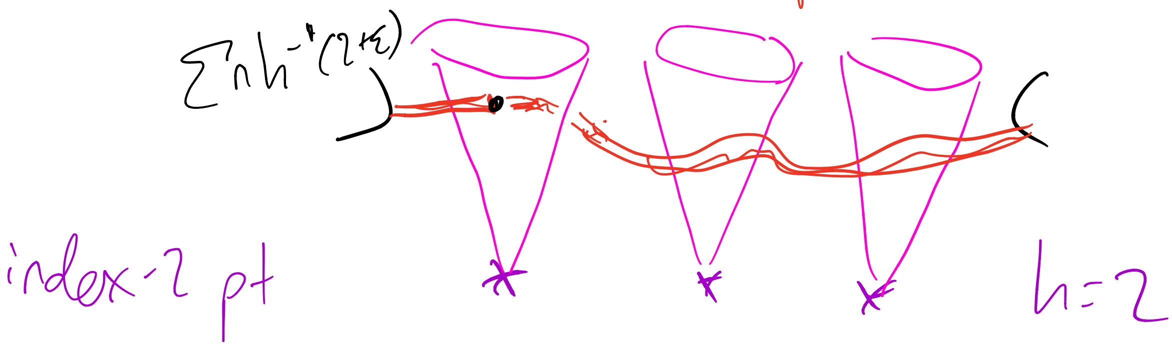
$\frac{3}{2} < t_1 < \dots < t_n < \frac{5}{2}$

so $h^{-1}(t_i) \cap \Sigma$ = link with one band

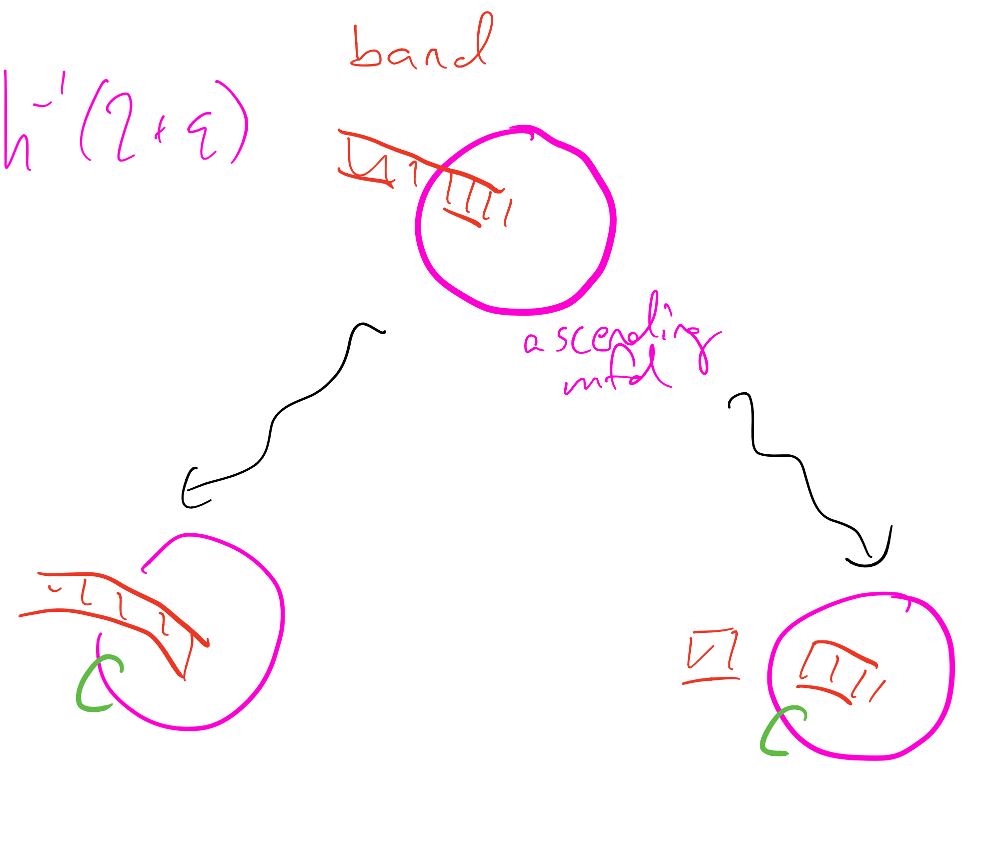


project (*) bands to $h^{-1}(\frac{3}{2})$ to find v (**)

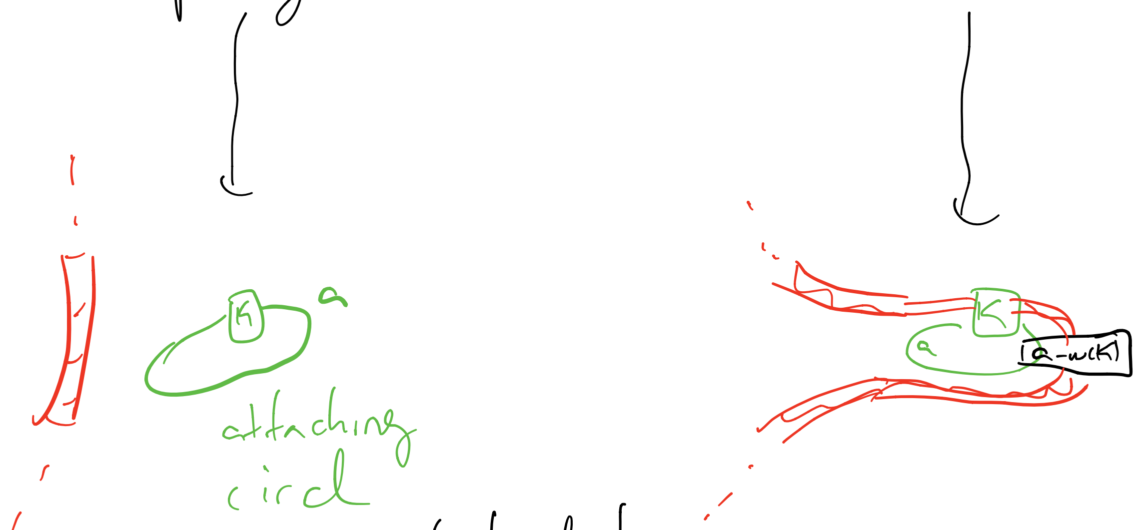
(*) Problem: How to project band to $h^{-1}(\frac{3}{2})$? Use $-\nabla h$ but get stuck if band intersects ascending mfd at index-2 crit pt of h



Have to make choice of how to push band off ascending mfd



In projection to $h^{-1}(\frac{3}{2})$, see

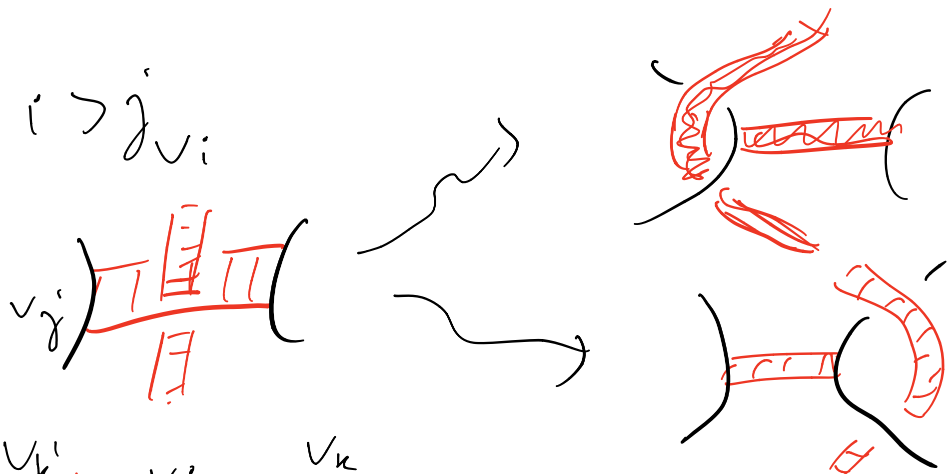


related by
2-handle band slide

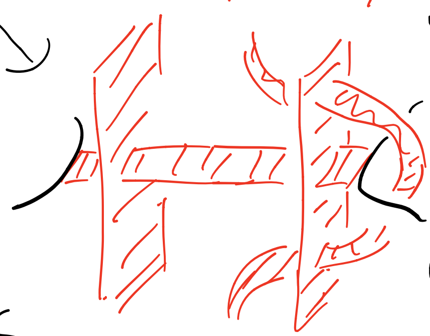
(**) need to make choices
so bands embedded in
$h^{-1}(\frac{3}{2})$ disjointly
and miss K circles
(and also $h^{-1}(\frac{3}{2}) \cap \Sigma$ avoids
K circles)

projections

Say v_i at height t_i
$i > j \Rightarrow v_i$ above v_j

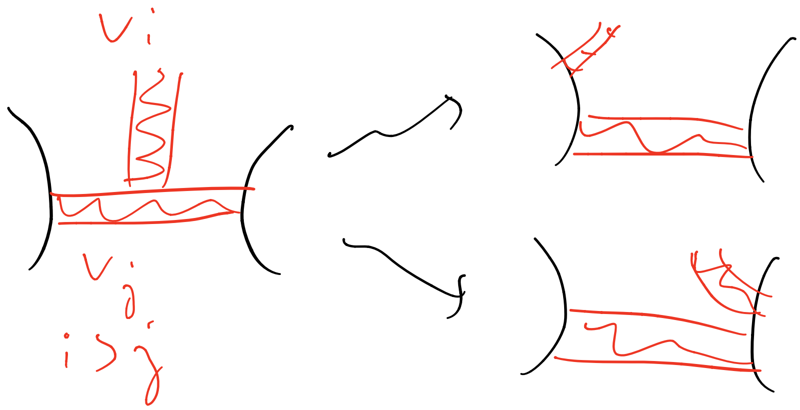


$i > j > k, k'$

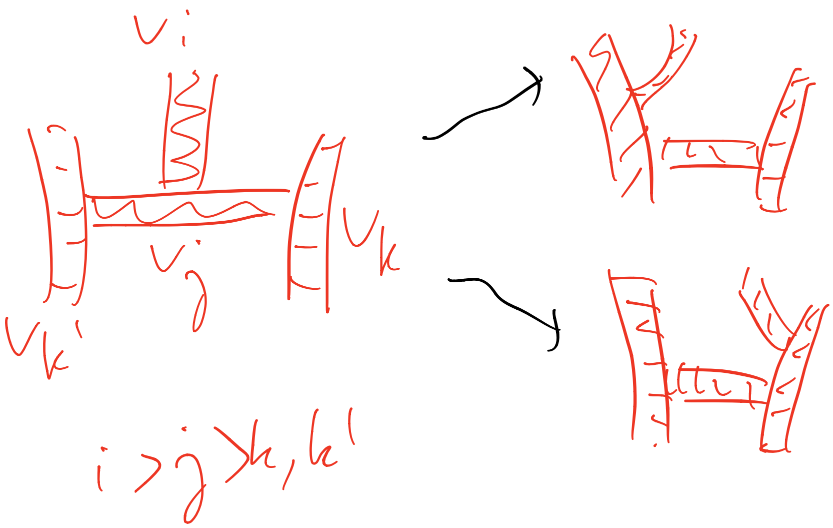


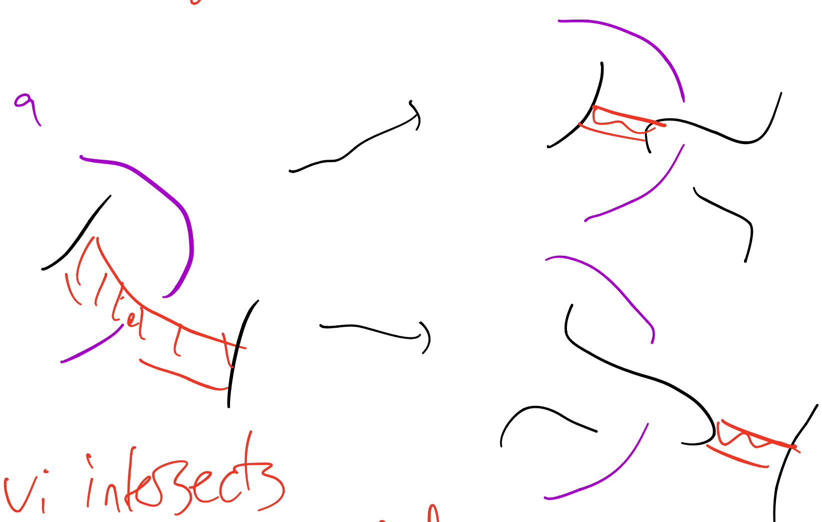
differ by band swims

removed ints of v_i with v_j, $k < j$, so repeat until remove ints of v_i with v_i



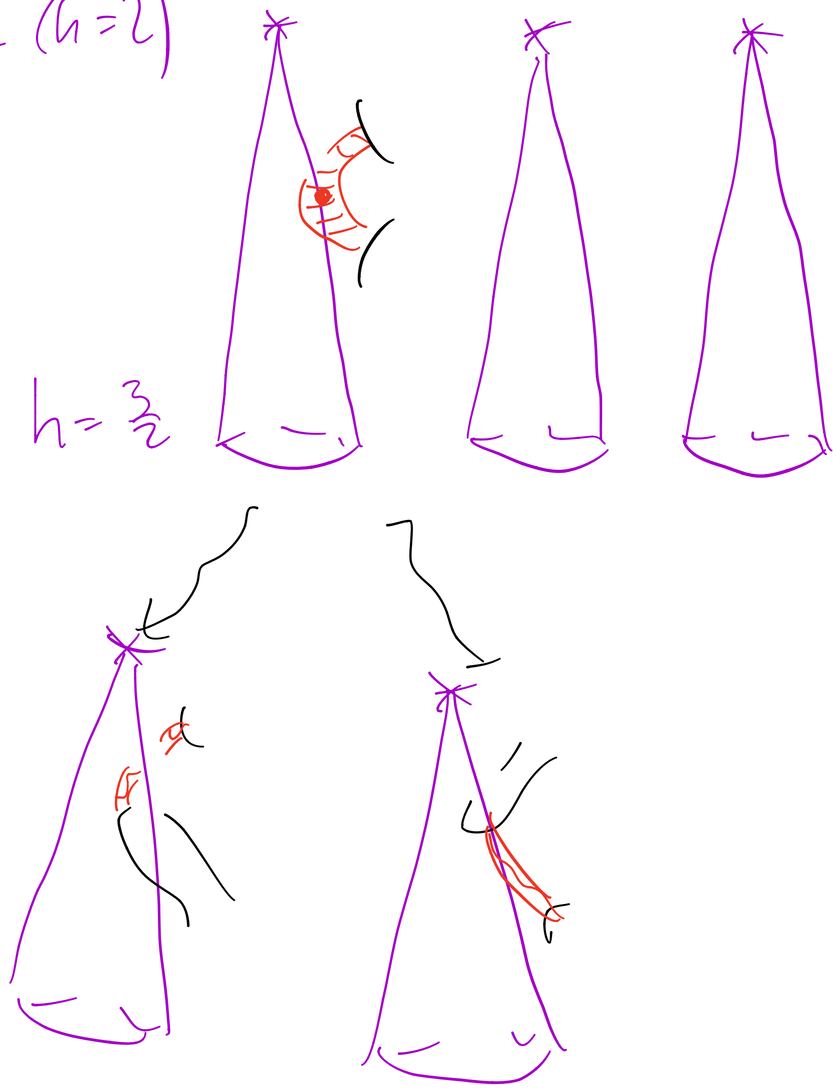
differ by band slide



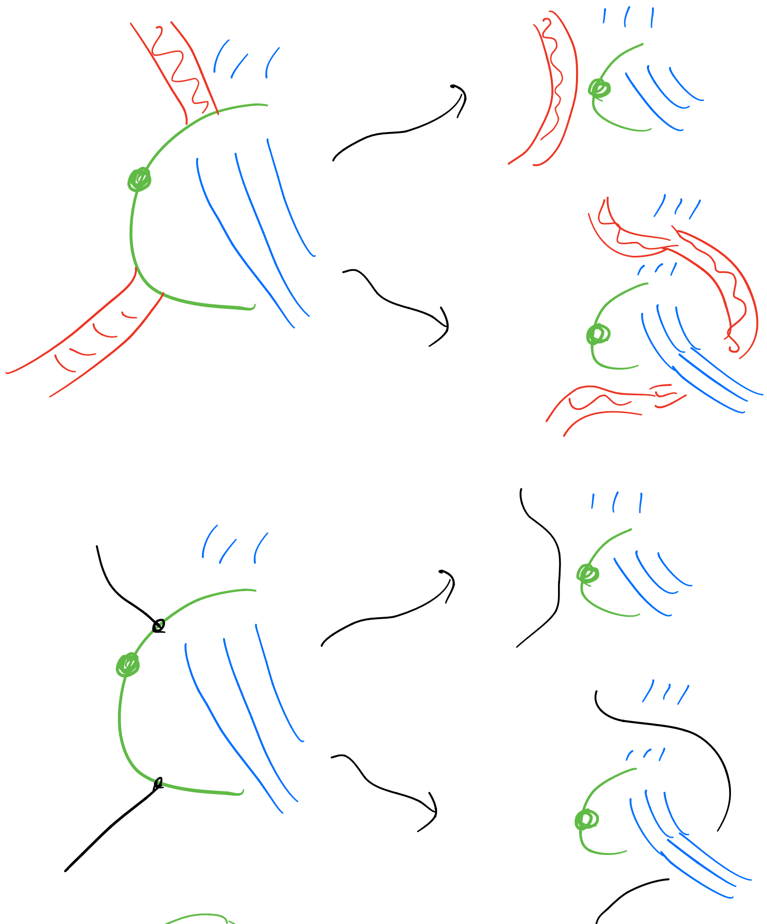


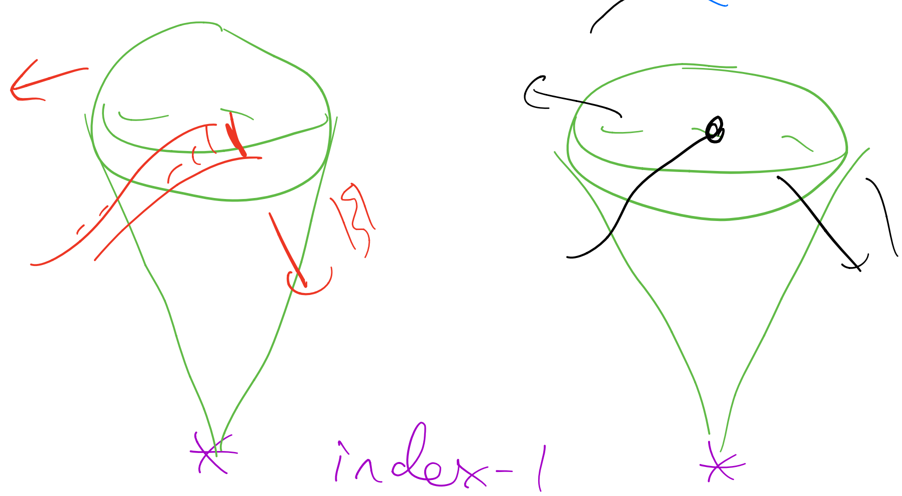
2-handle band swim

(v_i intersects descending mfd of index-2 pt index-2 (n=2)



dotted circle slide





Conclude:

- Σ has a diagram

Procedure above gives a diagram D_Σ well-defined up to band moves + isotopy

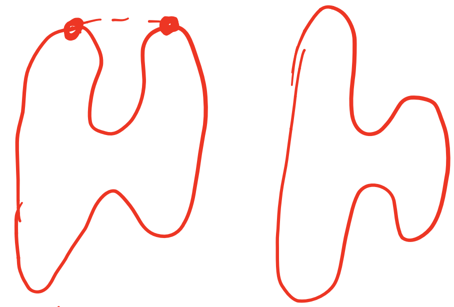
<u>Def</u>

$\Sigma \subset X^4$ <u>generic</u>

if
- Σ far from crit pts of h
- $h|_\Sigma$ Morse
- heights of crits of $h|_\Sigma$ all distinct

Lemma

If Σ_0, Σ_1 generic and isotopic through generic surfaces then $D_{\Sigma_0}, D_{\Sigma_1}$ related by band moves + isotopy



Pf Step 1 ⇝ put Σ_0, Σ_1 into horizontal-vertical position

→ Argue during isotopy, can keep Σ_t in hz position "vertical" (along $\pm\nabla h$) isotopy doesn't change projection to $h^{-1}(\frac{3}{2})$

"horizontal" (preserving $h|_\Sigma$) isotopy

Isotopes bands within cross-section

- Projections intersect
  $\rightsquigarrow$ band slide/swim
- Band intersects ascending mfd of index-2 crit
  $\rightsquigarrow$ 2-handle band slide
- Band intersects descending mfd of index-2 crit
  $\rightsquigarrow$ 2-handle band swim

If none of the above, projection changes by isotopy + dotted circle slides (i.e. isotopy in $h^{-1}(\frac{3}{2})$)

Nongeneric surfaces

(Kearton-Kurlin)

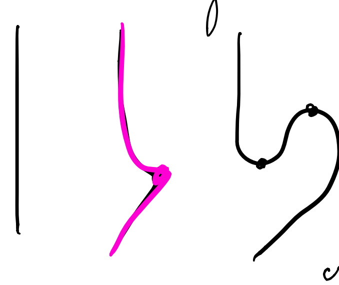
generic except ...

$A_1^+ A_1^+$ - sing two extrema @ same height

$A_1^+ A_1^-$ - sing extremum/band at same height

$A_1^- A_1^-$ - sing two bands @ same height

A_2 - sing local $x^2+y^3=0 \subset \mathbb{C}^2 = \mathbb{R}^4$

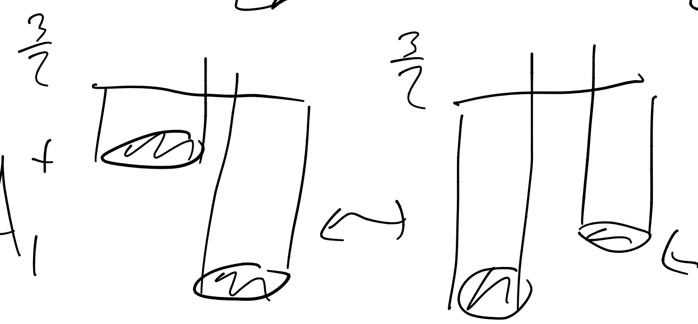


(birth or death of pair of $h|\Sigma$ crits)

Lemma

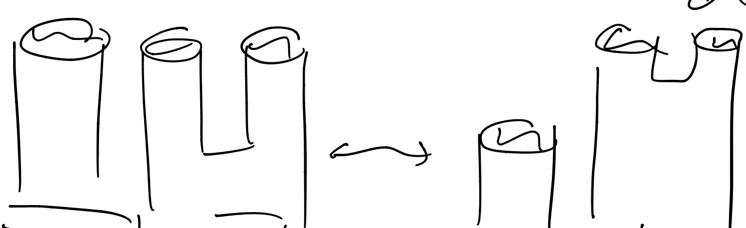
If Σ_0, Σ_1 generic isotopic through generic surfaces and $\boxed{1}$ sing as above, then D_{Σ_0} D_{Σ_1} related by band moves

$A_1^+ A_1^+$

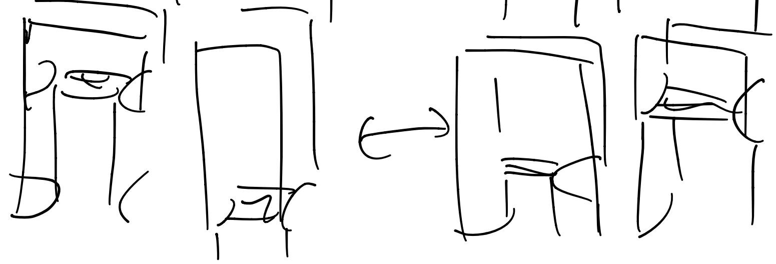


Doesn't affect diagram

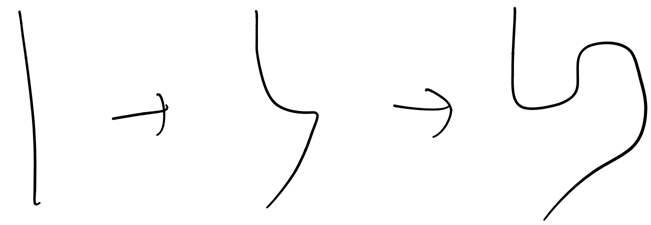
$A_1^+ A_1^-$



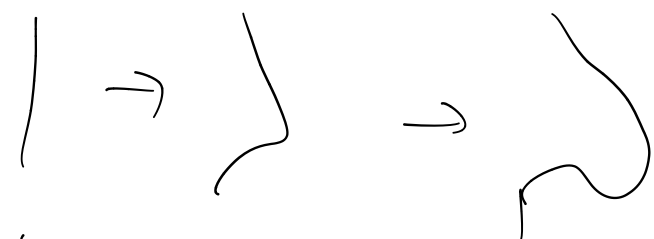
$A_1^- A_1^-$



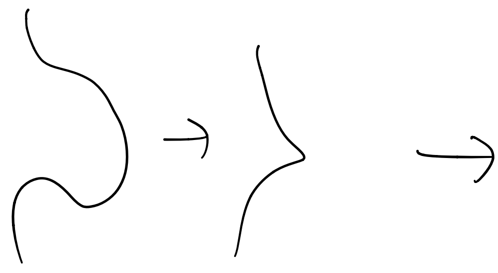
A_2



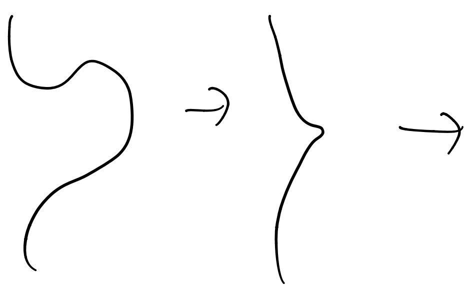
cap



cup



undo cup



undo cap

Lemma (Keerten-Kurlin (modified)
Thom (Jet spaces))

$CS = \{\text{surfaces} \hookrightarrow X^4\}$
Topology = Whitney topology

$X = \{A_1^{\pm}, A_1^{\pm}, A_2$ - singularities $\}$
$\overline{X}$ = codim-1 subspace of CS

$CS - \overline{X} = \{$ generic surfaces
+ surfaces intersecting
crit pts of $h \}$

$\therefore \Sigma_0, \Sigma_1$ isotopic (path in CS → take transverse to $\overline{X}$)
Can take isotopy through
generic surfaces + finitely many

$A_1^{\pm} A_1^{\pm}$, A_2 - singularities

$\rightsquigarrow \mathcal{D}_{\Sigma_0}$, $\mathcal{D}_{\Sigma_1}$ related by band moves. $\blacksquare$

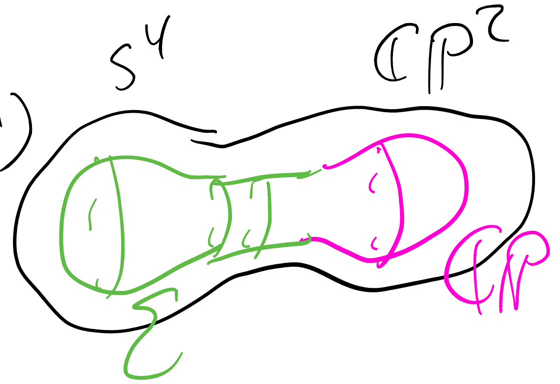
Application

Several spheres in $\mathbb{CP}^2$ representing $[\mathbb{CP}^1]$ can be shown to be isotopic to $\mathbb{CP}^1$

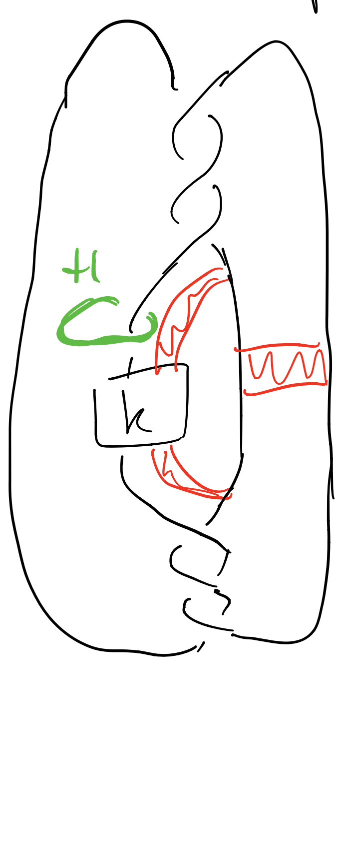
$\Sigma = S^2 \subset S^4$ $\Sigma \# \mathbb{CP}^1 = \mathcal{U}_\Sigma \subset \mathbb{CP}^2 = S^4 \# \mathbb{CP}^2$

Melvin

Diffeo $(\mathbb{CP}^2, \mathcal{U}_\Sigma) \cong (\mathbb{CP}^2, \mathbb{CP}^1)$
iff Gluck twist on $\Sigma \cong S^4$

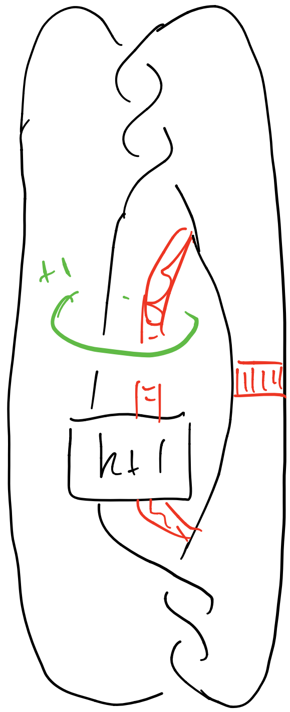


e.g. twist-spins

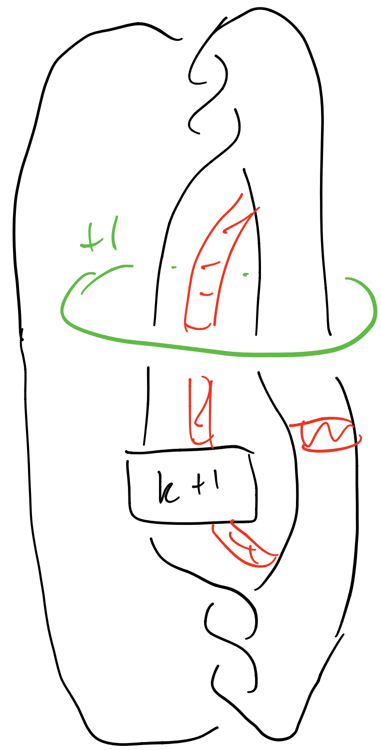


$k=1 \rightsquigarrow$ unknotted sphere $\# \mathbb{CP}^1$
$= \mathbb{CP}^1$

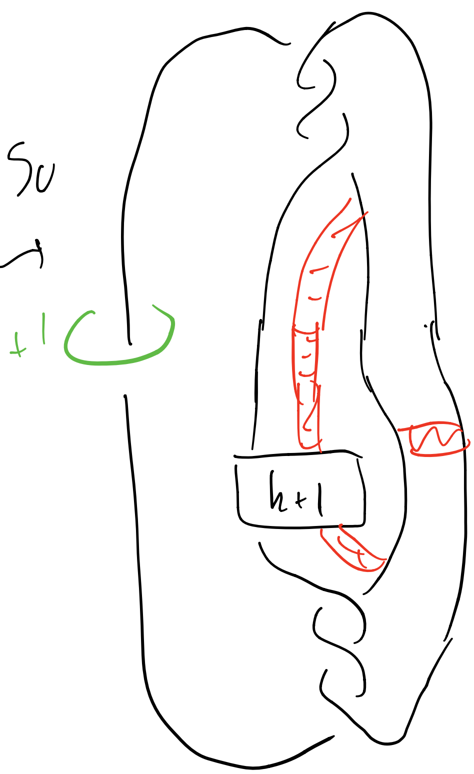
2-handle band slide

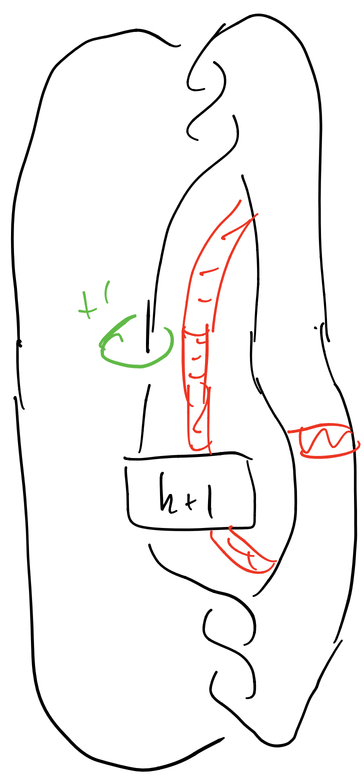


2-handle band swim

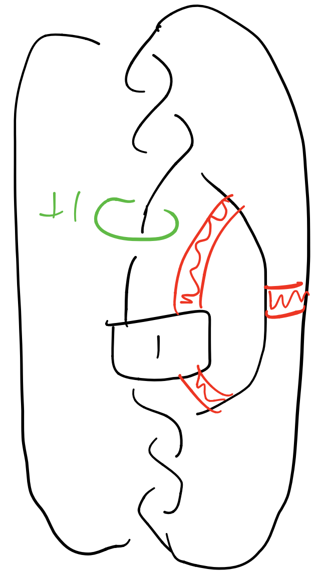


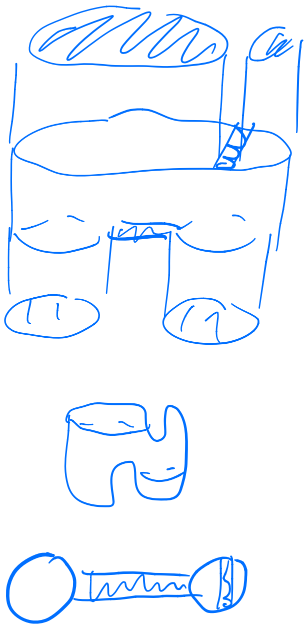
iso

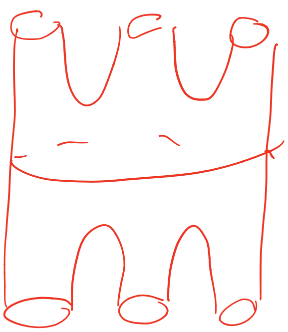


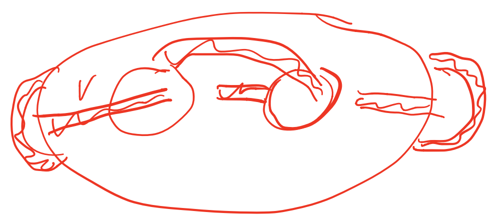


$\therefore$ U_ξ isotopic to

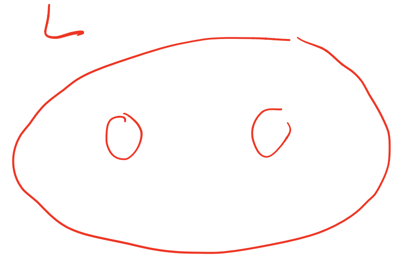






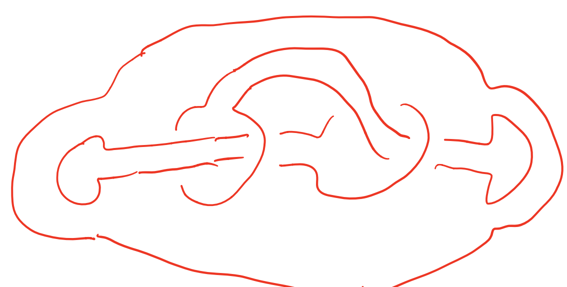


L



unlink

Lv



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