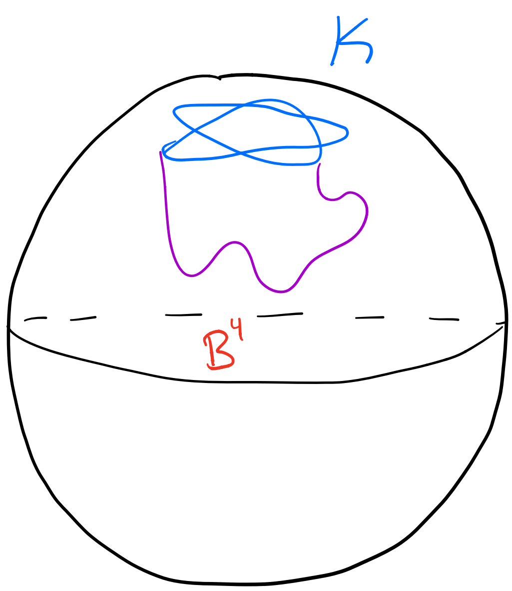
Equivalent characterizations of handle-ribbon knots
(Joint with Alex Zupan)

Outline:

1. Knots in dimension 3.5
2. Knot derivatives and Kauffman's conjecture
3. "Proof" of main theorem
4. Fibered knots and a theorem of Casson-Gordon

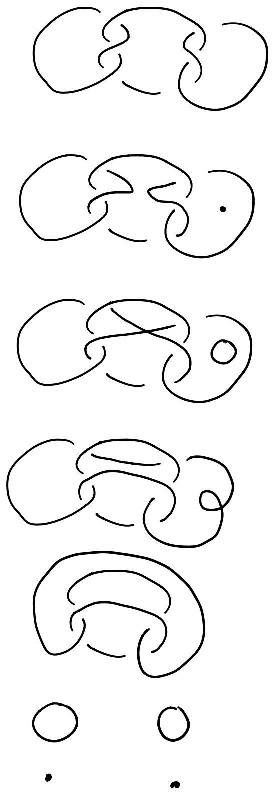
Theorem 1: A knot is handle-ribbon if and only if it admits an R-derivative

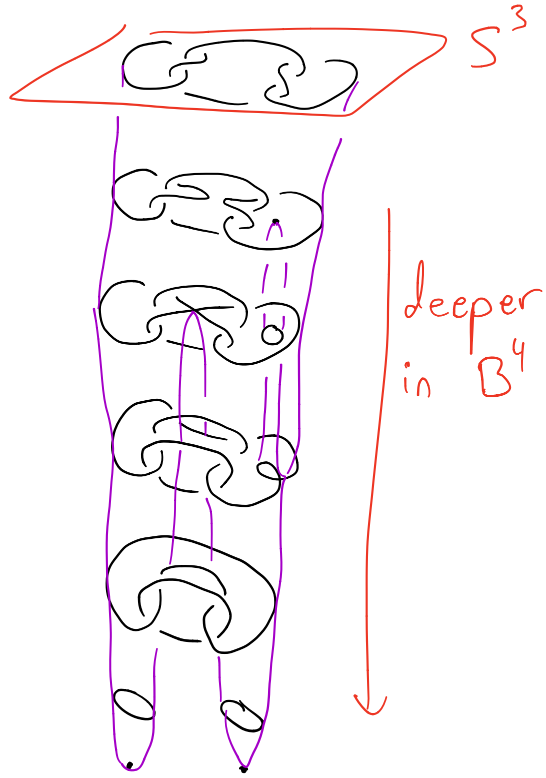
A knot K is an oriented circle in S^3.
We can view $S^3 = \partial B^4$ and study knots in 4D.



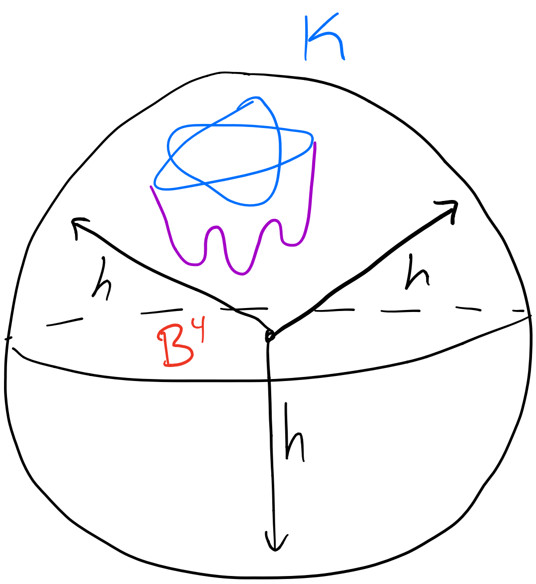
K is <u>slice in B^4</u> if K bounds a smoothly embedded disk in B^4.

Ex

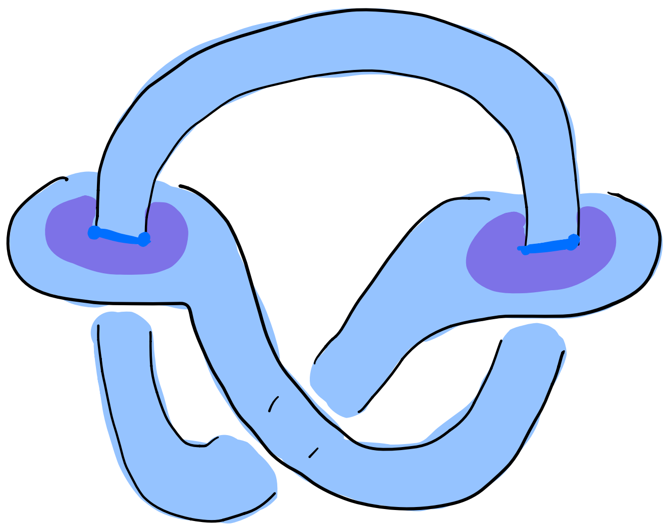




Nice case: K is ribbon if K bounds a slice disk D in B^4 so $h|_D$ is Morse with no maxima. (h = radial height)



This is nice because we can project to S^3 and get ribbon-immersed disk (or vice versa)



Ribbon-immersed disk in S^3:

self-intersections like

$\updownarrow$

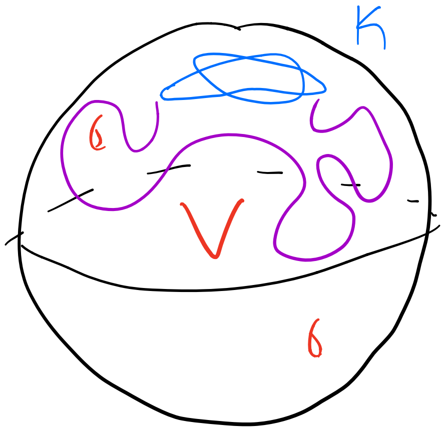
Ribbon disk in B^4: push the purple regions deeper

Relative 4D Poincaré Conjecture:

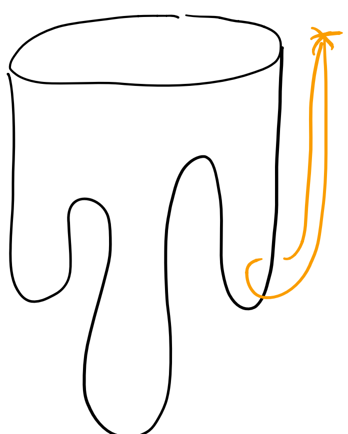
If V^4 is a homotopy 4-ball, then $V^4 \cong B^4$.

Now view $S^3 = \partial V^4$, V a homotopy 4-ball.

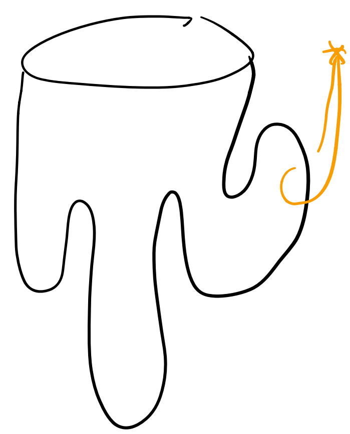
$K \subset S^3$ is homotopy-slice if K bounds a disk smoothly embedded in some V.



What does ribbon say about algebra?



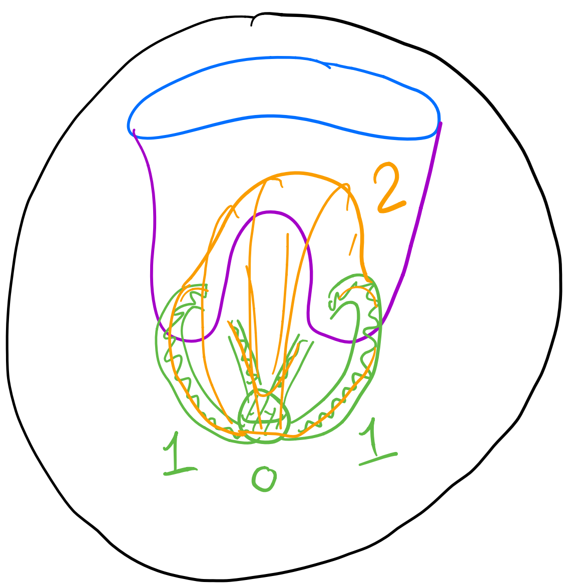
ribbon



not ribbon

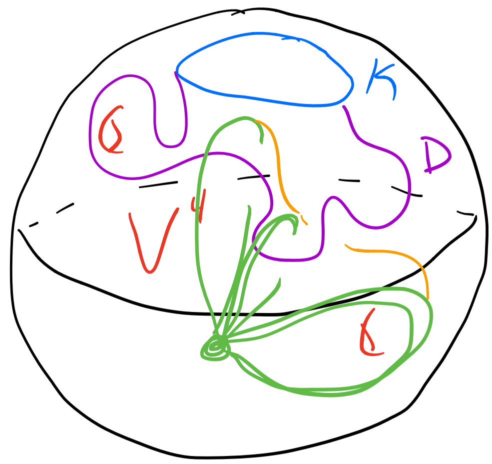
Def K is <u>homotopy-ribbon</u> if there exists a homotopy 4-ball V^4 and smooth disk $D \subset V$ with $\partial D = K$ and $i : \pi_1(S^3 \setminus K) \xrightarrow{\text{surjection}} \pi_1(V \setminus D)$

What does <u>ribbon</u> say about <u>topology</u>?



If D ribbon, $B^4 \setminus \nu(D)$ admits handle decomposition with no 3-handles.

Def K is <u>handle-ribbon</u> if there exists a homotopy 4-ball V and smooth disk $D \subset V$ with $(S^3, K) = \partial(V, D)$ so that $V^4 \setminus \nu(D)$ admits a handle decomposition with no 3-handles.



Inclusions

$$\underbrace{\{\text{Ribbon}\} \subseteq \left\{\begin{matrix}\text{Handle-ribbon} \\ \text{in } B^4\end{matrix}\right\} \subseteq \left\{\begin{matrix}\text{Homotopy-} \\ \text{ribbon in } B^4\end{matrix}\right\} \subseteq \{\text{slice}\}}$$

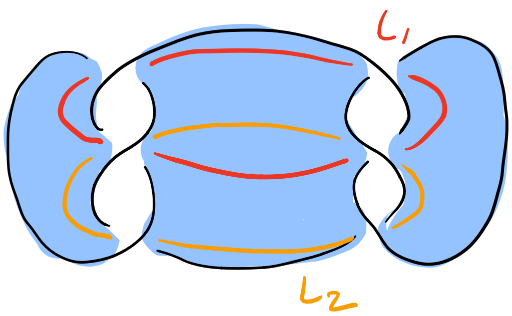
Slice-ribbon conjecture (Fox 1962): $\subseteq$ is $=$

$$\underbrace{\{\text{Handle-ribbon}\} \subseteq \{\text{Homotopy-ribbon}\} \subseteq \{\text{Homotopy-slice}\}}$$

Derivatives

A *derivative* for a knot K is a g-component link $L = L_1 \cup \dots \cup L_g$ on a genus-g Seifert surface F for K so that

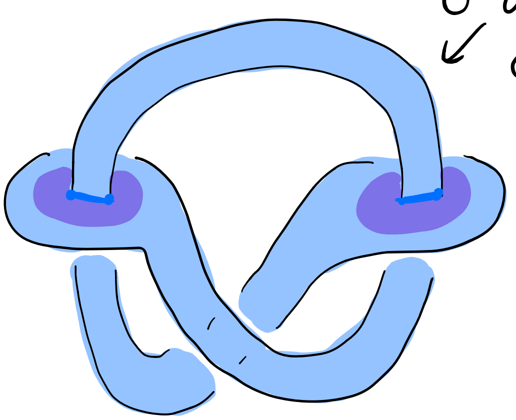
- $[L_i]$ independent
- $\ell k(L_i, L_j) = 0 \ \forall i \neq j$
- F induces 0-framing on $L_i \ \forall i$

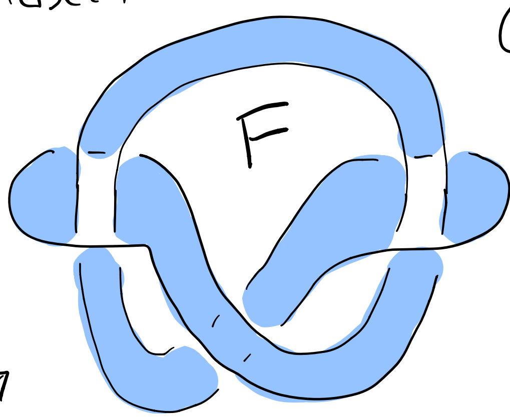


Note:
K has a derivative
$\iff$ K is algebraically slice

Characterizing ribbon knots

g intersections





Add genus at intersections to get Seifert surface

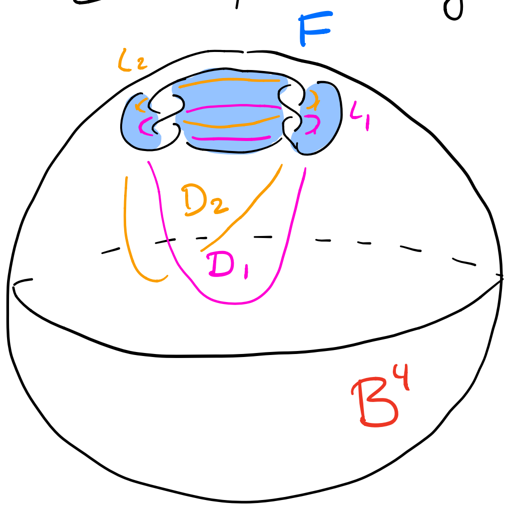
genus-g

Get a g-component unlink on F that is a derivative

K ribbon $\iff$ K has an unlink derivative

Slice knots

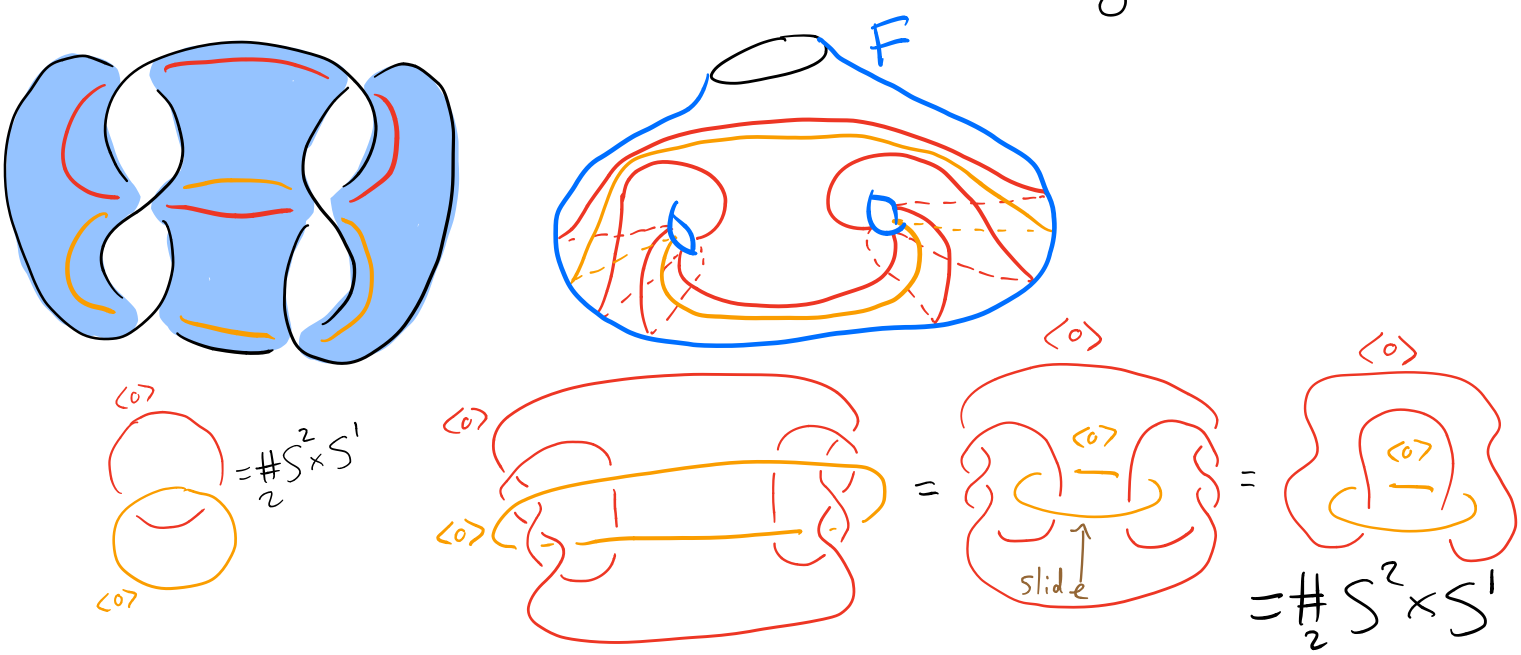
A *slice derivative* for K is a derivative $L = L_1 \cup \cdots \cup L_g$ that bounds $\bigsqcup_g D^2$ in B^4.



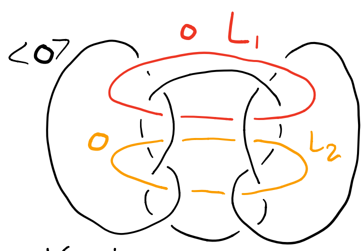
$F \overset{\text{compress}}{\underset{\text{along } \sqcup D^2}{\rightsquigarrow}}$ a slice disk for K

K has a slice derivative $\Longrightarrow$ K is slice

An *R-derivative* for K is a derivative $L = L_1 \cup \cdots \cup L_g$ with $S^3_0(L) \cong \#_g S^2 \times S^1$

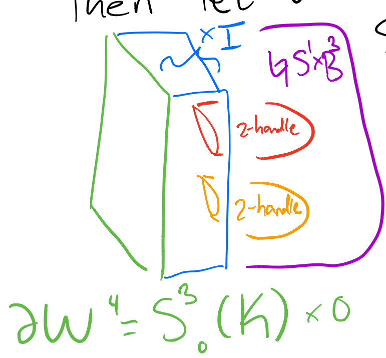


If K has an R-link derivative L,



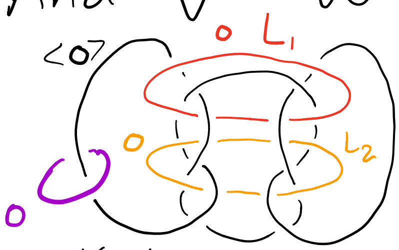
Kirby diagram

Then let W^4 be



$S^3_0(K) \times I$
$\cup$ 0-framed 2-handles along $L \times 1$
$\cup \natural_g S^1 \times B^3$

And $V^4 = W^4 \cup$ 0-framed 2-handle attached to $\mu(K \times 0)$



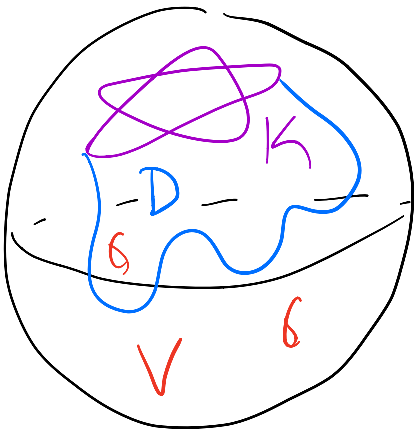
Kirby diagram

Conclude: If K has R-derivative, then K is handle-ribbon.

Thm (M-Zupan)

A knot K has an R-derivative if and only if K is handle-ribbon.

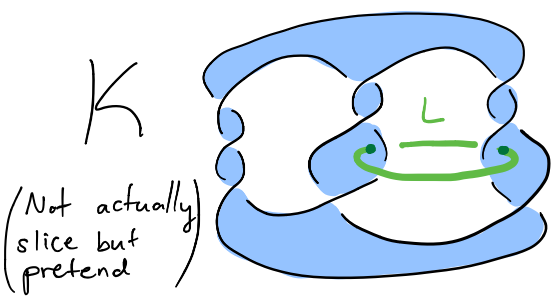
If K is handle-ribbon, then



$$V \setminus \nu(D) = S^3_0(K) \times I$$
$$\cup (2\text{-}, 3\text{-}, \text{and } 4\text{-handles})$$

L = attaching circles of 2-handles

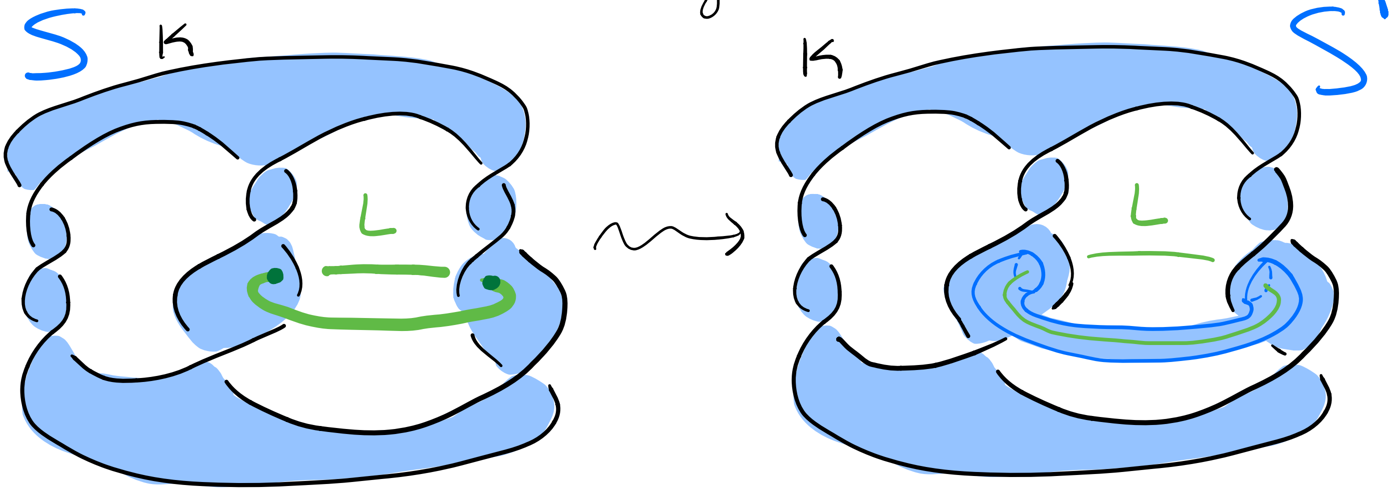
$$\rightsquigarrow S^3_0(K \cup L) = \#_{|L|+1} S^2 \times S^1$$



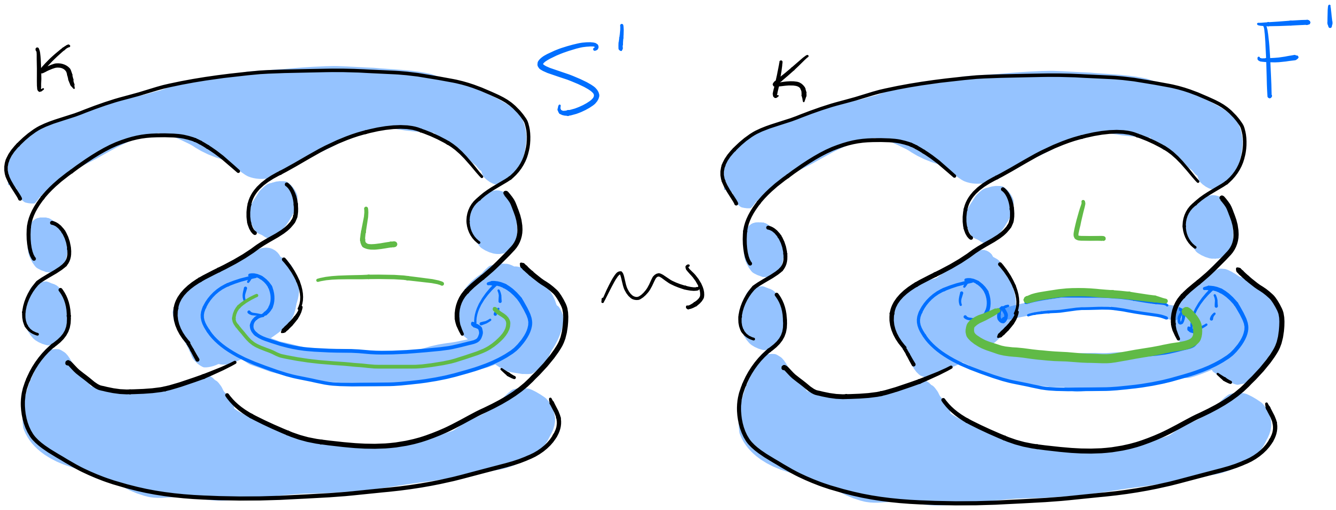
S a Seifert surface for K

$\langle L, S \rangle = 0$

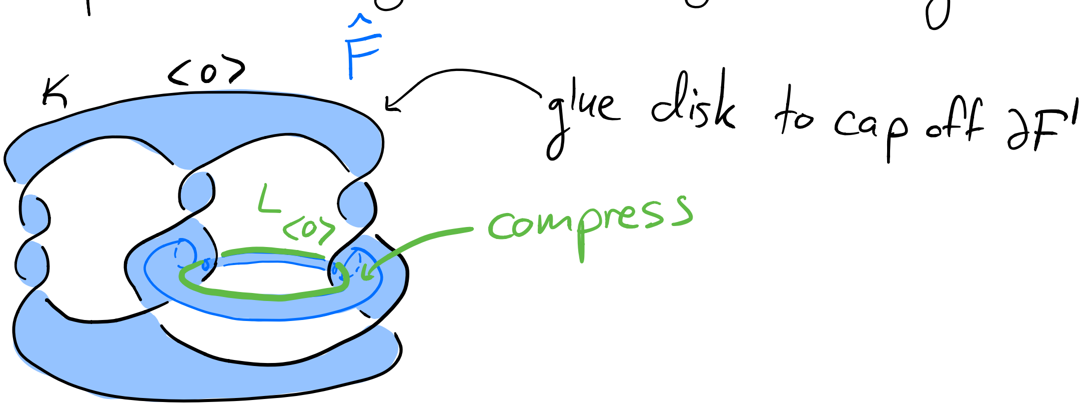
Add tubes to S to get S' disjoint from L.

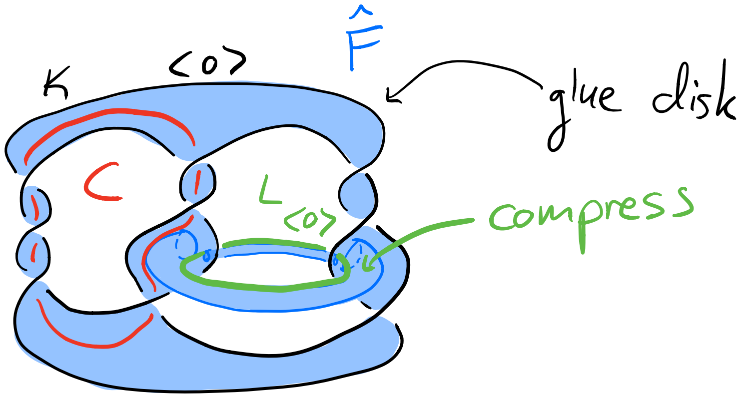


Now add more tubes $S' \rightsquigarrow F'$ so that L lies on S''.



Say $g(F) = |L| + n$. If $n=0$, we're done.
In $S^3_0(K \cup L) = \#_{|L|+1} S^2 \times S^1$, cap off F' and compress along L to get a genus-n surface $\hat{F}$.





If $n > 0$, then $\hat{F}$ is compressible along some curve C.

Since C = unknot in $S^3_0(K \cup L)$

- C 0-framed by F
- $lk(C, L_i) = 0 \;\forall i$

Since $[C] \neq 0$ in $H_1(\hat{F})$, $[C]$ independent from $[L_i]$ in $H_1(F')$

Set $L := L \cup C$ and continue inductively. $\blacksquare$

Related theorem of Casson-Gordon (1983)

Say K is fibered: $S^3 \setminus \nu(K) \cong F \times_\varphi S^1$
$$\downarrow$$
$$S^1$$

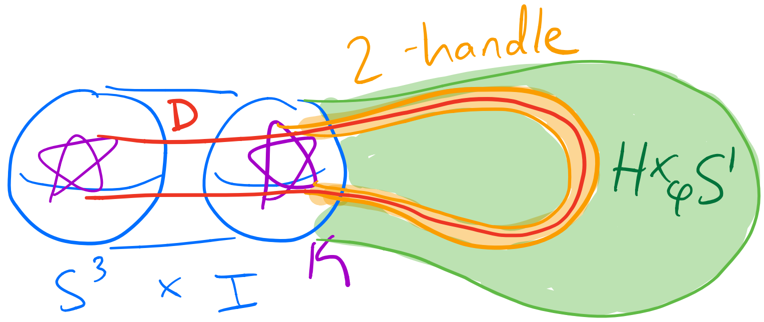
Then K is homotopy-ribbon if and only if φ extends to a handlebody.

Cor For K fibered:
K is homotopy-ribbon if and only if there exists a homotopy 4-ball V^4 and smooth disk $D \subset V$ s.t.

- $(S^3, K) = \partial(V, D)$
- $V \setminus \nu(D)$ is a bundle of handlebodies over S^1 ($\downarrow S^1$)
(Write "D is fibered")

Pf $\Leftarrow$ Write down presentation for π_1

$\Rightarrow$ Start with K fibered, homtopy-ribbon.
$CG \Rightarrow \varphi$ extends to $\hat{\varphi}: H \to H$



$V = S^3 \times I \cup$ (0-framed 2-handle along $K \times 1$) $\cup H \times_{\hat{\varphi}} S^1$
$D = K \times I \cup$ (core of 2-handle)

Thm (M-Zupan)

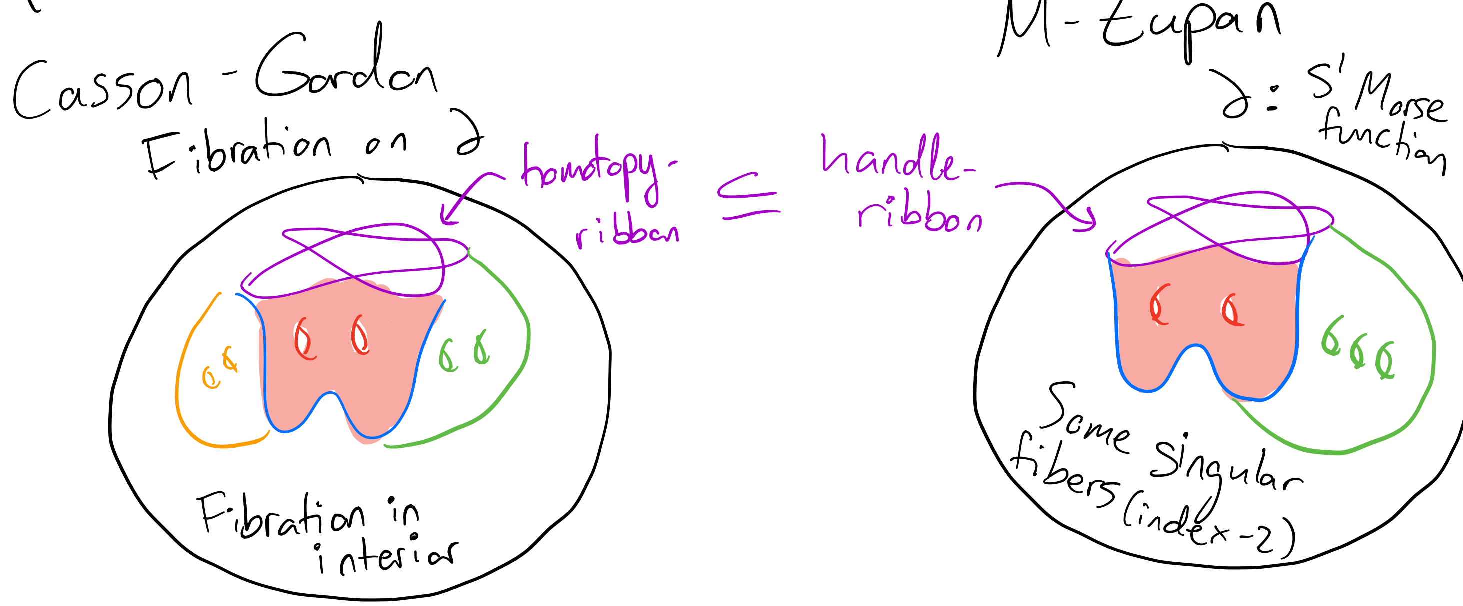
K a knot. Then K is <u>handle-ribbon</u> if and only if there exists a circular Morse function $f: S^3_0(K) \to S^1$

$$\begin{cases} 0 & \text{index-0 pts} \\ n & \text{index-1 pts} \\ n & \text{index-2 pts} \\ 0 & \text{index-3 pts} \end{cases}$$

extending to a circular Morse function $\hat{f}: W^4 \to S^1$ s.t.

- $\hat{f}$ has n interior critical points, all index-2
- regular fibers of $\hat{f}$ are handlebodies,
- $\chi(W^4) = 0$.

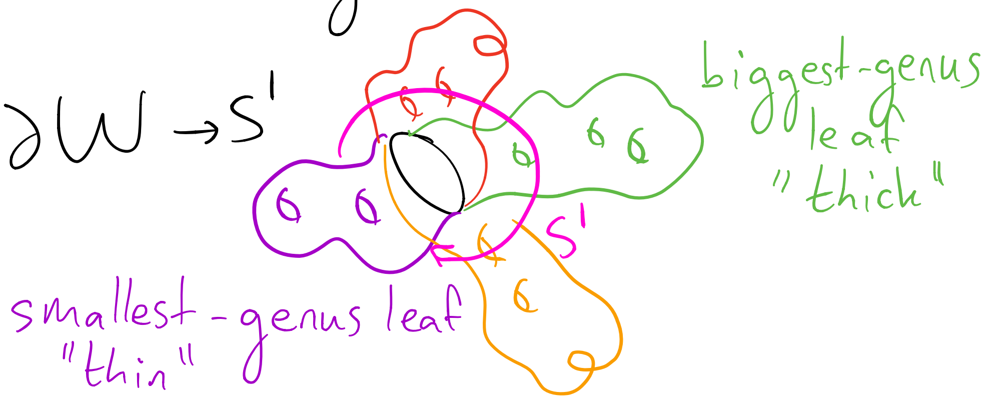
(There is actually a stronger theorem in terms of Morse 2-functions, but I didn't want to introduce them.)

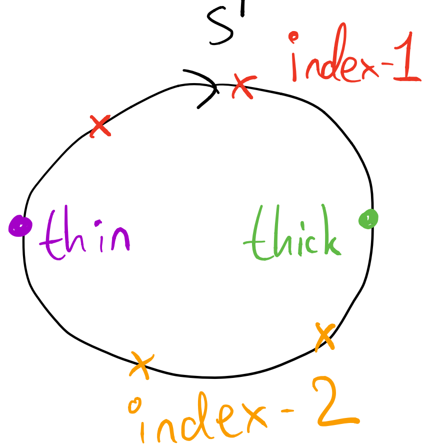


Pf (easy direction only)

$\Leftarrow$ Starting with $\hat{f}: W \to S^1$

$f: \partial W \to S^1$



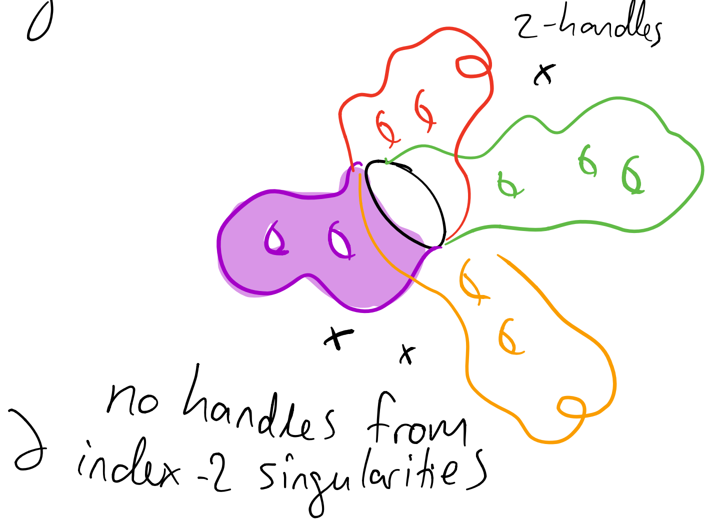


W = thin handlebody ← 0, 1-handles → $\hat{f}^{-1}(0)$

∪ 1-handles for ∂ singularities → $\hat{f}^{-1}[0,\pi]$

∪ 2-handles for interior singularities = $\hat{f}^{-1}[0, 2\pi-\varepsilon]$

∪ 1-, 2-handles to glue $\hat{f}^{-1}(2\pi-\varepsilon)$ to $\hat{f}^{-1}(0)$.



W has no 3-handles!

Also #1-handles = #2-handles + 1.

$\Rightarrow H_*(W;\mathbb{Z}) = H_*(S^1;\mathbb{Z})$

Moreover $\pi_1(W) = \langle\langle \mu(K) \rangle\rangle$

$\rightsquigarrow$ Glue 2-handle to $\mu(K)$ and get V;

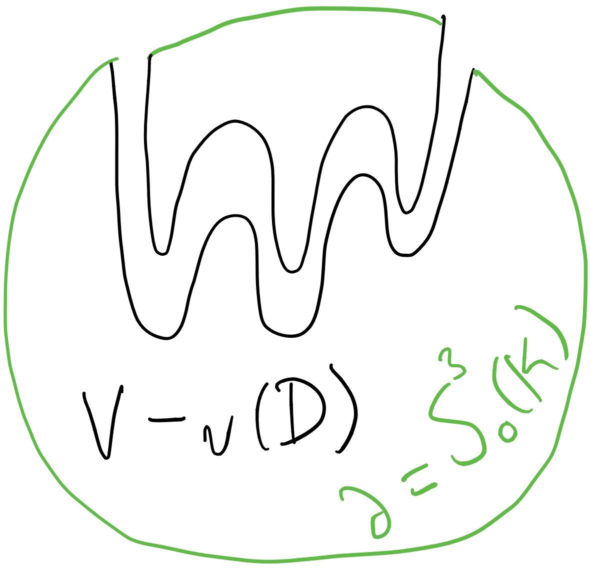
D = cocore of 2-handle.

$(S^3, K) = \partial(V, D) \quad V \setminus \nu(D) = W.$

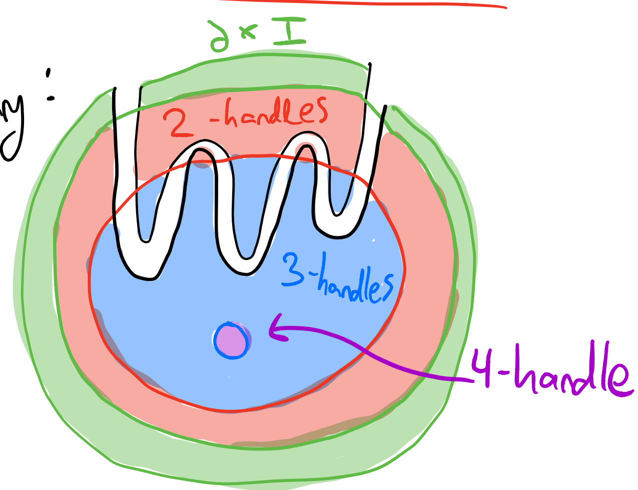
(Not a proof) Hard direction

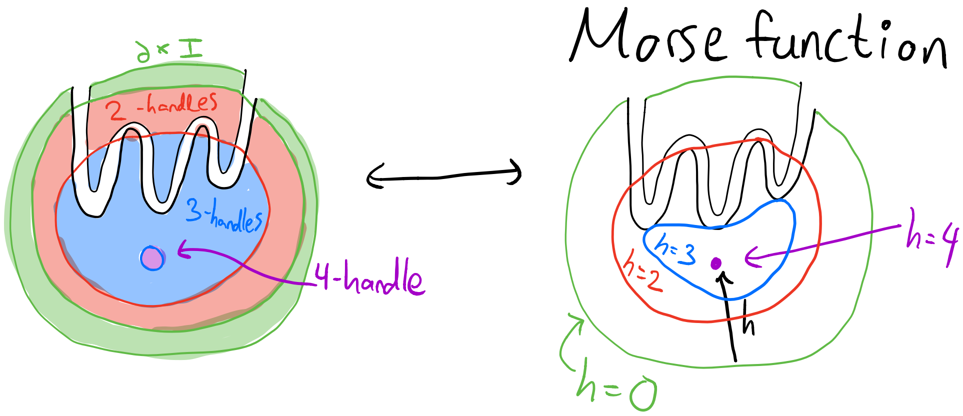
(Really would prove via Morse 2-functions)

K handle-ribbon



rel boundary:





- Started with $f_0 := f : h^{-1}(0) \to S^1$
- Explicitly build $f_t : h^{-1}(t) \to S^1$ so
  $F(x) := f_{h(x)}(x) : V \setminus \nu(D) \to S^1$ is smooth
- Use Cerf theory to control
  $\{\text{critical pts of } F\} \subseteq \{(x,t) \mid f_t(x) \text{ not Morse}\}$

Thanks :)