Knot cobordisms, torsion in Floer homology, bridge index

Joint w/ András Juhász + Ian Zemke

Overarching idea

Given cobordism Σ from K to J

$S^3 \times I$

use topology + embedding of Σ to relate torsion in e.g. $HFK^-(K)$, $HFK^-(J)$

(or conversely use torsion to constrain topology + embedding of Σ)

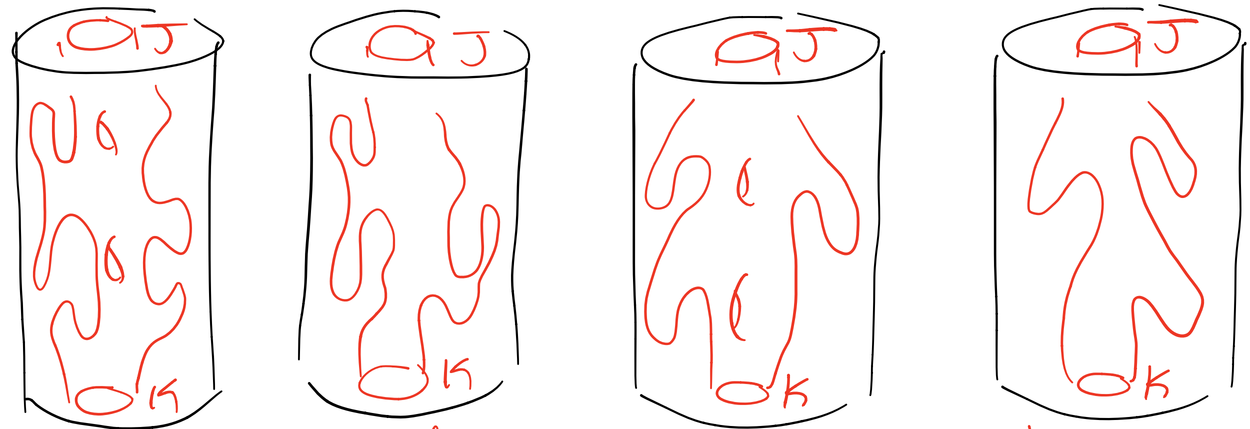
Background K, J oriented links

Cobordism from K to J is a CONNECTED surface $\Sigma \underset{\text{proper}}{\overset{\text{smooth}}{\hookrightarrow}} S^3 \times I$ so

$$\partial \Sigma = (K \times 0) \cup (J \times 1)$$

- When $\Sigma \cong$ annulus, say K, J are <u>concordant</u>

concordance = symmetric relation



cobordism concordance ribbon cobordism ribbon concordance

- When $h|_{\Sigma}$ Morse with no maxima,
  $h := \mathrm{proj}_I$
  say Σ is a <u>ribbon cobordism</u>.
  If Σ also annulus, then
  a <u>ribbon concordance</u>.

  Note ribbon cobordism/concordance
  is <u>not</u> symmetric.

<u>Def</u> If U concordant to J, then J
unknot knot
<u>slice</u>.

If $\exists$ ribbon concordance from U to J,
then J is <u>ribbon</u>.

Longstanding open question (Fox 1962)
Is every slice knot ribbon?

Yes for 2-bridge knots (Lisca 2007)
Yes for odd 3-stranded pretzels (Greene + Jabuka 2007)

In general ???

Maybe false, but difficult to obstruct ribbon concordance.

Zemke 2019
Knot Floer homology can obstruct ribbon concordance

If Σ ribbon concordance from K to J, then Σ induces injection really more general

$$\widehat{HFK}(K) \hookrightarrow \widehat{HFK}(J)$$

(Rank, map preserves gradings)

Recovers

Gordon: $\deg \Delta(K) \leq \deg \Delta(J)$

If J also ribbon concordant to K, then $\Delta(K) = \Delta(J)$

Rmk, Gordon conjectured if K, J both ribbon concordant to each other then $K = J$ (isotopic)

Zemke $\Rightarrow \widehat{HFK}(K) \cong \widehat{HFK}(J)$

Levine + Zemke $\Rightarrow$ and $Kh(K) \cong Kh(J)$

Inspired many more papers shortly after

Levine + Zemke

Σ also induces injection

$Kh(K) \hookrightarrow Kh(J)$

(Khovanov homology)

M + Zemke

can weaken "ribbon"

Sarkar Defined "ribbon distance"
$d(K, J)$ when K, J concordant

smallest n s.t. $\exists$

sequence of ribbon or inverse ribbon concordances connecting K to J, each with at most n max/min.

$n = 3$

and gave lower bound from Khovanov via torsion in a certain perturbation

Lidman Vela-Vick Wong
Extend to setting of ribbon homology cobordisms

Knot Floer homology

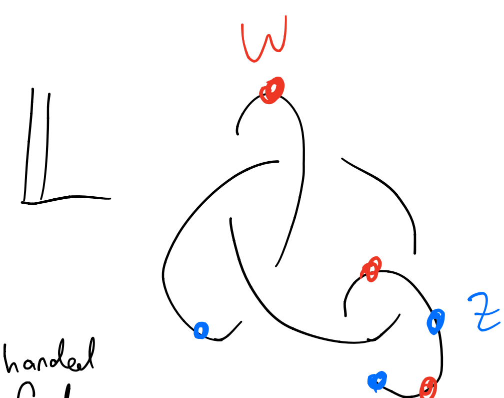
multi-based link

$\mathbb{L} = (L, w, z)$

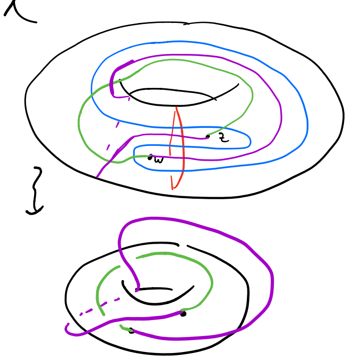
L = oriented link

w, z basepts on L

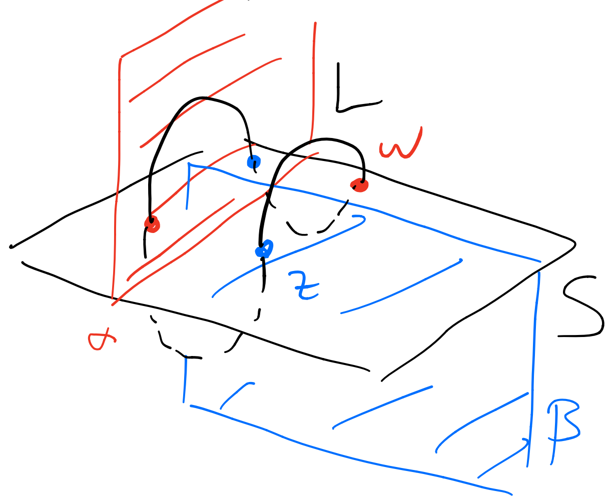
- At least one w + one z on each component of L
- alternate around L



Left-handed Trefoil



Heegaard $\rightsquigarrow$ diagram (S, α, β, w, z) for $\mathbb{L}$



Build chain complex generated by pts of intersection of

$$\mathbb{T}_\alpha \cap \mathbb{T}_\beta \subset \mathrm{Sym}^n(S)$$

$$\begin{array}{ccc} \uparrow & & \uparrow \\ \alpha_1 \times \cdots \times \alpha_n & & \beta_1 \times \cdots \times \beta_n \end{array}$$

choice of differential

$\hat{\partial}$: counts pseudo hol disks in Sym^n which miss $w \cup z$

∂^- : counts p hol disks in Sym^n which miss w, weights z with variable

$$\partial^- x = \sum_{y \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta} \sum_{\substack{\phi \in \pi_2(x,y) \\ \mu(\phi)=1 \\ n_w(\phi)=0}} (\#\mathcal{M}(\phi)/\mathbb{R}) v^{n_z(\phi)} \cdot y$$

Extend over $\mathbb{F}_2[v]$ equivariantly.

(This is probably not helpful if you are not familiar with Knot Floer homology. HFK^- can be taken so far as a black box [I recommend Manolescu's notes for actual exposition on this]. For experts, this has just established which conventions are being used.)

Now $HFK^-(K)$ is a finitely generated module over polynomial ring $\mathbb{F}_2[\overset{\text{counts } z\text{-grading}}{v}]$. $HFK^-(K)$ decomposes (non-canonically) as

$$\mathbb{F}_2[v] \oplus \underbrace{HFK^-_{red}(K)}_{\text{the } \mathbb{F}_2[v] \text{ torsion submodule of } HFK^-(K).}$$

So define for knot $K \subset S^3$

$Ord_v(K) :=$

$\min\{n \in \mathbb{N} \cup \{0\} \,/\, v^n \cdot HFK^-_{red}(K) = 0\}$

<u>Rmk</u>

Always $Ord_v(K) < \infty$

$Ord_v(K) = 0 \Leftrightarrow K = \text{unknot}$

$Ord_v(T_{p,q}) = p-1$ for $0 < p < q$ coprime

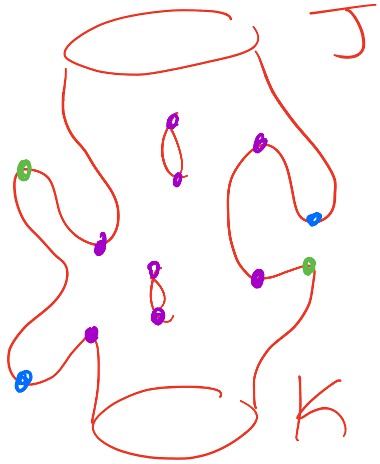
$Ord_v(K) = Ord_v(\overline{K})$

Main thm (JMZ)

Σ cobordism from K to J

so $h|_\Sigma$ Morse with m minima, b saddles, and some # maxima.

$$Ord_v(K) \leq \max\{b-m, Ord_v(J) - \chi(\Sigma)\}$$
$$= (b-m) + \max\{0, Ord_v(J) - M\}$$



maxima

b saddles

m minima

In this schematic
$b-m = 6$
$\chi(\Sigma) = -4$
$\therefore$
$\mathrm{Ord}_v(K) \leq \max\{6, \mathrm{Ord}_v(J) + 4\}$

ribbon cobordism: $M = 0$
$b - m = -\chi(\Sigma) = 2g(\Sigma)$
$\mathrm{Ord}_v(K) \leq \mathrm{Ord}_v(J) + 2g(\Sigma)$
$\mathrm{Ord}_v(J) \geq \mathrm{Ord}_v(K) - 2g(\Sigma)$

Corollary

$$\mathrm{br}(K) \not\geq \mathrm{Ord}_v(K)$$
(bridge index)

Pf

Ord_v not additive under #
-- takes max value

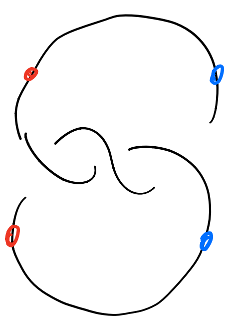
$$\mathrm{Ord}_v(A \# B) = \max(\mathrm{Ord}_v(A), \mathrm{Ord}_v(B))$$

(using the fact that $HFK^-(K)$ is finitely generated module)

$\therefore$ So $\mathrm{Ord}_v(K) = \mathrm{Ord}_v(K \# \bar{K})$

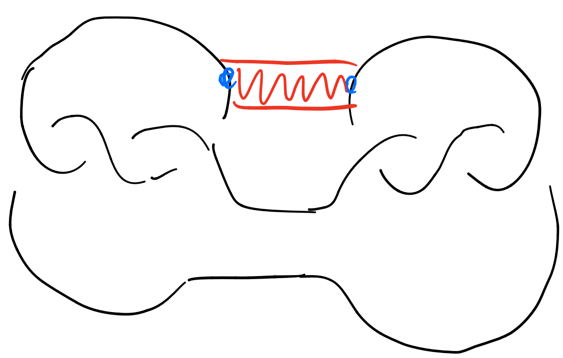
$\exists$ ribbon concordance from U to $K \# \bar{K}$ with $br(K)-1$ saddles (and minima)

K

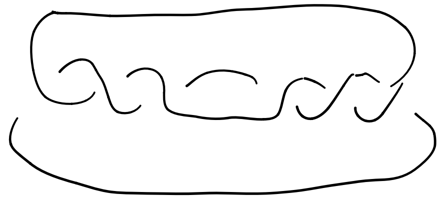


$br(K) = 2$

$K \# \bar{K}$



$\rightsquigarrow$ surger



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Thm says

$$\mathrm{Ord}_v(U) \leq \max\{0\ \ , \mathrm{Ord}_v(K)\}$$

Not interesting. But:

Turn concordance upside down ($br(K)-1$ saddles and maxima)

$$\mathrm{Ord}_v(K) \leq \max\{br(K)-1, \mathrm{Ord}_v(U)\}$$
$$= br(K)-1.$$

(<u>Rmk</u>, Sharp for torus knots.)

<u>Another Corollary</u>

because (upside down)
$\mathrm{Ord}_v(J) \leq \max\{b-M, \mathrm{Ord}_v(K)\}$
$\rightsquigarrow \underbrace{b-M}_{=m} \geq \mathrm{Ord}_v(J)$
$0 = b-M-m$

If Σ a ribbon concordance from K to J with b saddles then

either $b \leq \mathrm{Ord}_v(J) = \mathrm{Ord}_v(K)$

or $b \geq \mathrm{Ord}_v(J) \geq \mathrm{Ord}_v(K)$

(Note already know $\mathrm{Ord}_v(J) \geq \mathrm{Ord}_v(K)$ by Zemke)

If Σ ribbon cobordism from K to J of genus-g ($2g+n$ saddles, n minima)

then $\mathrm{Ord}_v(J) \le \mathrm{Ord}_v(K) + 2g$

(Refined)

Cobordism Distance

$$d(K,J) = \min\{ \underbrace{\max\{b-m, b-M\}}_{\text{lowest for ribbons}} \mid \text{cobordisms from } K \text{ to } J \}$$

(where $m = \#$minima, $b = \#$saddles, $M = \#$maxima)

Thm $\Rightarrow$ $d(K,J) \ge |\mathrm{Ord}_v(K) - \mathrm{Ord}_v(J)|$

Pf $d = d(K,J) \rightsquigarrow \mathrm{Ord}_v(K) \le \underbrace{\mathrm{Ord}_v(J)}_{\ge 0} + \underbrace{b-m}_{\ge -\chi_\Sigma}$

so $\mathrm{Ord}_v(K) \le \mathrm{Ord}_v(J) + d$

similarly $\mathrm{Ord}_v(J) \le \mathrm{Ord}_v(K) + b - M \le \mathrm{Ord}_v(K) + d$

$\rightsquigarrow |\mathrm{Ord}_v(K) - \mathrm{Ord}_v(J)| \le d$

Finally

torsion distance

$$d_t(K,J) = \min\{ d \in \mathbb{N} \mid v^d HFK^-(K) \cong v^d HFK^-(J) \}$$

$\le d_r(K,J)$ (Sarkar's ribbon distance)

$d_t(K,U) = \mathrm{Ord}_U(K)$

can be large even for slice knots, e.g. $T_{p,q} \# \overline{T_{p,q}}$

$\left(\mathrm{Ord}_U = p-1,\ 0 < p < q\right)$

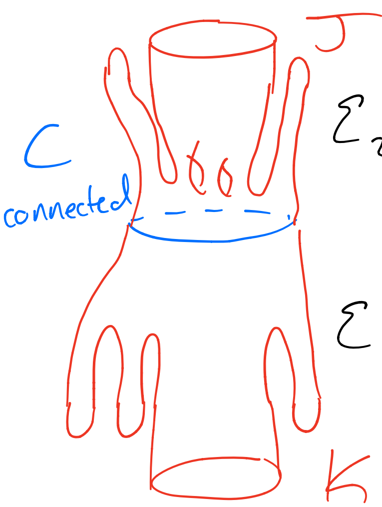
$\rightsquigarrow d_r(T_{p,q} \# \overline{T_{p,q}}, U) \geq p-1$

can be arbitrarily large.

Proof of Theorem (In talk, prove Zemke original theorem first)

Σ cobordism from K to J

Reorder crits of $h|_\Sigma$ so



Σ_2 $\begin{cases} M \text{ maxima} \\ b-m \text{ saddles} \end{cases}$

Σ_1 $\begin{cases} m \text{ saddles} \\ m \text{ minima} \end{cases}$

(standard normal form argument)

Σ_1: K to C ribbon concordance

Σ_2: C to J upside-down ribbon cobordism

Decorate Σ = w-region + z-region

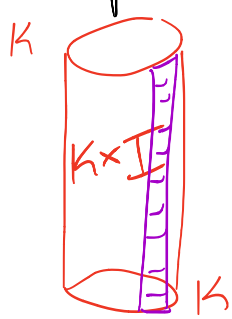
So K, J have w, z basepts in correct region

Zemke + Juhász

many papers, some joint

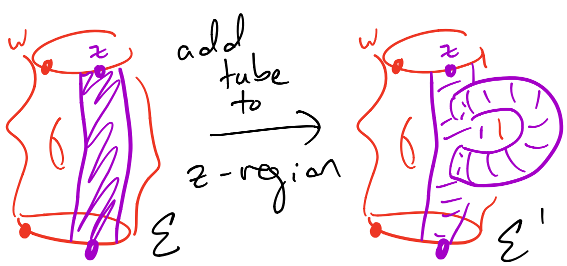
decorations induce map $\mathcal{F}_\Sigma$ on HFK^-

In particular:

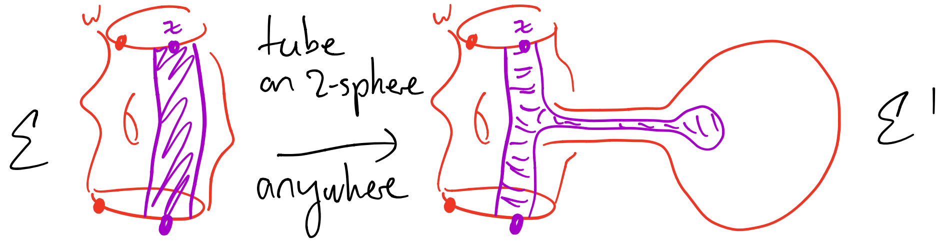


If $\Sigma = K \times I$, decoration vertical

$\mathcal{F}_\Sigma = id$



$\mathcal{F}_\Sigma$ $\qquad$ $\mathcal{F}_{\Sigma'} = v\,\mathcal{F}_\Sigma$



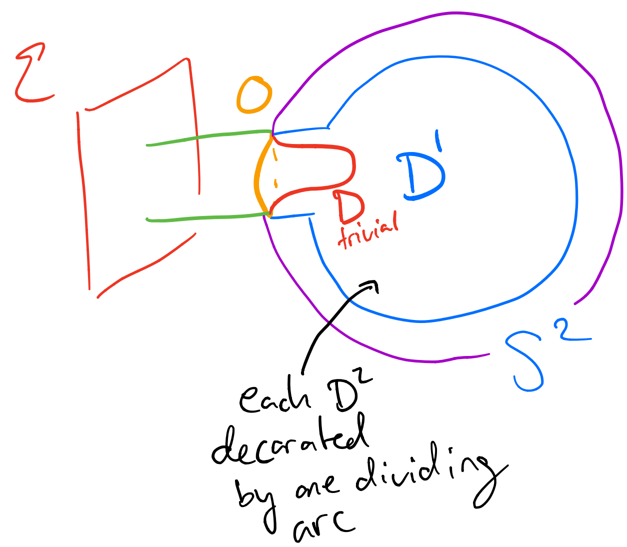
F_Σ $\qquad$ $F_{\Sigma'} = F_\Sigma$ (Zemke crucial observation)

Because:

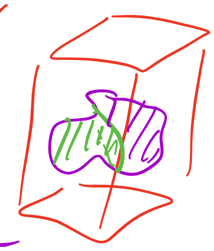
(Note Juhász-Marengon 2016 for $\widehat{HFK}$)

If Σ = 2-sphere divided into 2 disks

then $F_\Sigma = id$ (map for closed surface divided into 2 parts by one curve only depends on genus of each piece)



$K = J = \phi$



$W = \nu(S^2)$

$\cong S^2 \times D^2$

$O \subset \partial S^2 \times D^2$

$= \partial D' = \partial D$

$\exists$ unique spin^c str t_0 on W whose Chern class evaluates trivially on $\{0\} \times S^2$

$t_0' = t_0|_{S^1 \times S^2}$

Claim

$\mathcal{F}_{\underset{\hat{W}}{D}, t_0} = \mathcal{F}_{D', t_0}$ as maps from

$HFL^-(\phi)$ to $HFL^-(\underset{S^1\times S^2}{\hat{O}}, t_0^!)$

Pf both maps nonzero

(since $\mathcal{F}_{S^2 \subset S^4} = id$ factors)

have same gradings

$(gr_z = -\frac{1}{2})$

$CFL^-(O \subset S^1 \times S^2, t_0^!)$

$\cong \mathbb{F}_{2\,(-\frac{1}{2})} \oplus \mathbb{F}_{2\,(\frac{1}{2})} \otimes_{\mathbb{F}_2} \mathbb{F}_2[v]$

no "room" -- grading information determines the map

Back to theorem proof

Say $S = \Sigma \cup \overline{\Sigma}$ = cobordism from K to K

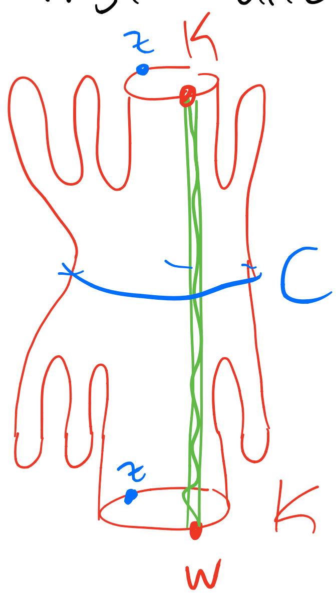
Pick two base pts on K $(z+w)$ far from projections of crits of $h|_\Sigma$

take w-region to be narrow/vertical, mirror in $\overline{\Sigma}$

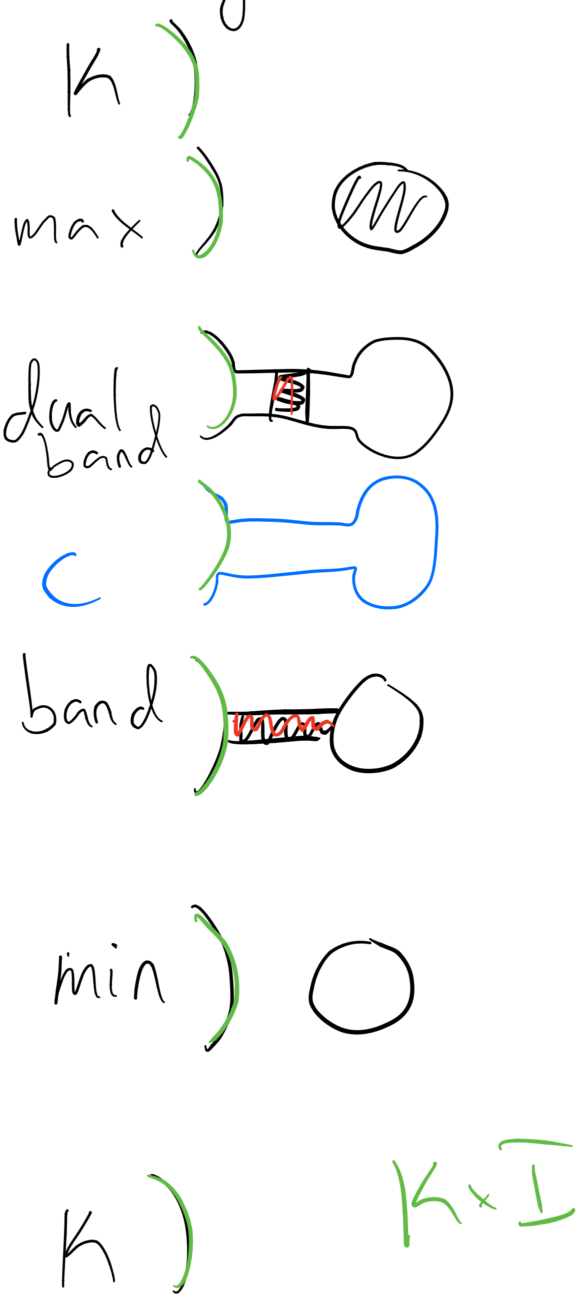
convention in these notes is $A \cup B$ (left A, right B) means first (bottom) A then (top) B

Therefore $F_{A \cup B} = F_B \circ F_A$

First understand $\Sigma_1 \cup \bar{\Sigma}_1$



moving up see



tubed on spheres to $K \times I$

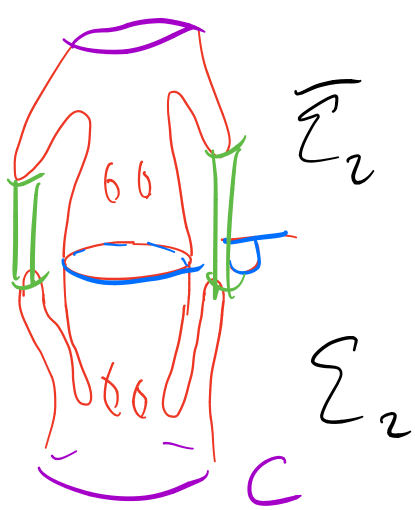
$\rightsquigarrow F_{\Sigma_1 \cup \overline{\Sigma}_1} = id$

So F_{Σ_1} injection

(This is Zemke's original proof!)

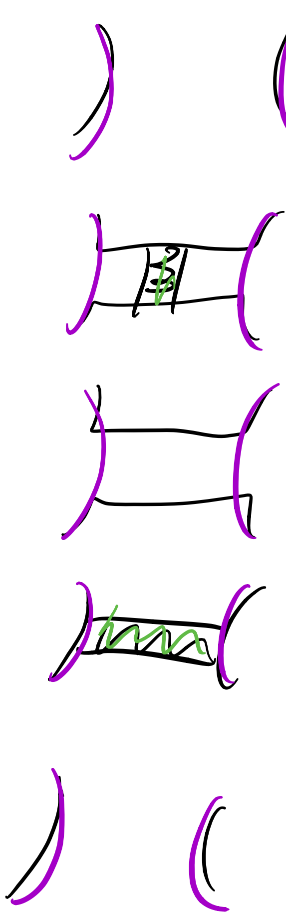
$\rightsquigarrow \mathrm{Ord}_v(C) \geq \mathrm{Ord}_v(K)$

Now understand $\Sigma_2 \cup \overline{\Sigma}_2$



- Add M tubes to connect maxima to dual minima (Now only bands)

Now movie looks like sequence of

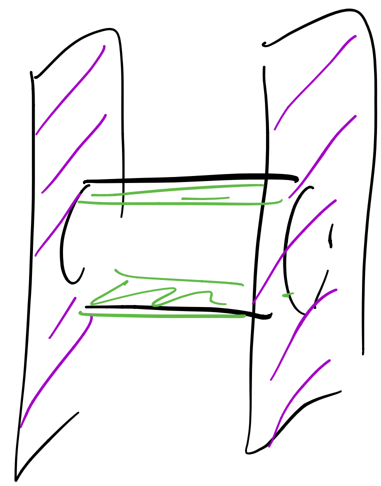


dual

band

$C \times I$

$\rightsquigarrow$



tubes attached to $C \times I$ (in z-region)

Specifically, b-m tubes

so $\mathcal{F}_{\Sigma_2 \cup \overline{\Sigma}_2 + \text{tubes}} = V^{b-m}\, \mathrm{id}_{HFK^-(C)}$

also $\mathcal{F}_{\Sigma_2 \cup \overline{\Sigma}_2 + \text{tubes}} = V^{M}\, \mathcal{F}_{\Sigma_2 \cup \overline{\Sigma}_2}$

$$V^M \mathcal{F}_{\Sigma_2 \cup \bar{\Sigma}_2} = V^{b-m}\, \mathrm{id}_{HFK^-(C)}$$

So
$$\mathcal{F}_{\Sigma_2 \cup \bar{\Sigma}_2}(V^{a+M} x) = V^{a+b-m} x \qquad a \geq 0$$

Note $\mathcal{F}_{\Sigma_2 \cup \bar{\Sigma}_2}$ factors through $HFK^-(J)$.

If $a \geq \max\{0, \mathrm{Ord}_V(J) - M\}$

then $V^{a+b-m} x = 0$ (because of factoring)

$\therefore$ $\mathrm{Ord}_V(C) - (b-m) \leq \max\{0, \mathrm{Ord}_V(J) - M\}$

$$\mathrm{Ord}_V(K) \leq b - m + \max\{0, \mathrm{Ord}_V(J) - M\}$$

$$= \max\{b-m,\ \mathrm{Ord}_v(J)-\chi(\Sigma)\}$$

$(\chi(\Sigma) = m-b+M)$. $\blacksquare$

(End of talk)

Continued notes

d_r = Sarkar's ribbon distance

$d_r(K,J) =$

$\min\{d \in \mathbb{N} \cup \{0\} \mid \exists$ ribbon concordances to or from

$K = K_0$ to/from K_1 to/from K_2 $\cdots$ $K_n = J$ which each have at most d minima$\}$

(say $d_r = \infty$ if undefined)

Properties

(1) $d_r(K, J) < \infty \iff K, J$ concordant

(2) $d_r(K, J) = 0 \iff K, J$ isotopic

(3) $d_r(K, J) = d_r(J, K)$

(4) $d_r(K, J) \leq \max(d_r(K, A), d_r(A, J))$

(3)+(4) $\rightsquigarrow$ d_r is a metric

(2) $\rightsquigarrow$ d_r reasonable

(1) $\rightsquigarrow$ d_r refinement of sliceness

Thm (JMZ)

If $d = d_r(K, J) < \infty$

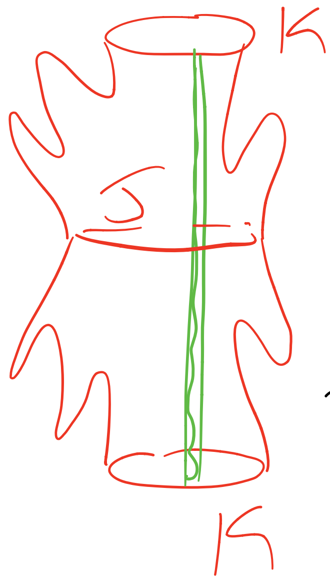
then $v^d HFK^-(K) \cong v^d HFK^-(J)$

Pf Enough to show if $\exists$ ribbon concordance from K to J

with n minima, then

$$v^n HFK^-(K) \cong v^n HFK^-(J).$$

Same idea: Consider $\Sigma \cup \overline{\Sigma}$, decorated with vertical narrow w subregion



$\rightsquigarrow$ again by Zemke, get

$$F_{\Sigma \cup \overline{\Sigma}} = id_{HFK^-(K)}$$

Then attach n tubes to $\overline{\Sigma} \cup \Sigma$ (other order of concatenation!)

$\rightsquigarrow$ get

$$v^{\overset{\text{M of }\Sigma}{n}} \bar{\mathcal{F}}_{\bar\Sigma \cup \Sigma} = v^{\overset{\text{b-m of }\bar\Sigma}{n}} \mathrm{id}_{HFK^-(J)}$$

So element $\in v^n HFK^-(J)$

$x v^{a+n}$ $(a \geq 0)$

$$v^n \bar{\mathcal{F}}(x v^{a+n}) = x v^{a+2n}$$

$$\bar{\mathcal{F}}(x v^{a+n}) = x v^{a+n}$$

$$\therefore \quad \bar{\mathcal{F}}_{\Sigma \cup \Sigma}\Big|_{v^n HFK^-(J)} = \mathrm{id}_{v^n HFK^-(J)}$$

factors through
$v^n HFK^-(K)$

and $\mathcal{F}_{\Sigma \cup \bar{\Sigma}}\Big|_{v^n HFK^-(K)} = \mathrm{id}_{v^n HFK^-(K)}$

factors through $v^n HFK^-(J)$,

so $v^n HFK^-(K) \cong v^n HFK^-(J)$. $\blacksquare$

Rmk

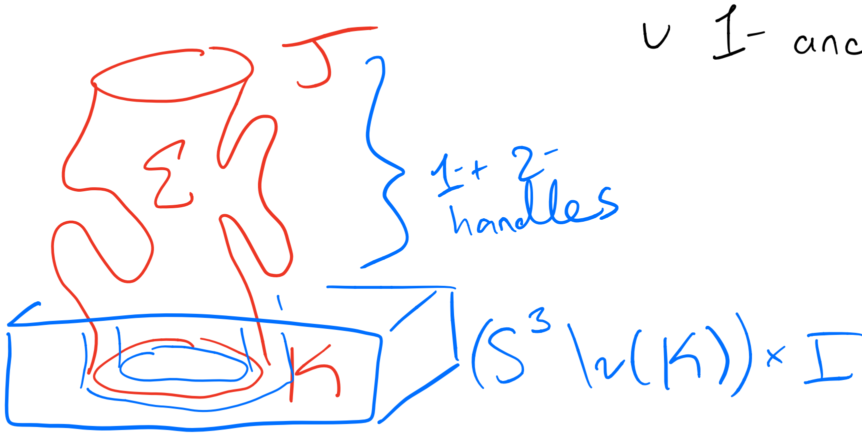
$$\Rightarrow d_r(T_{p,q} \# \overline{T_{p,q}}, U) \geq d_t(T_{p,q} \# \overline{T_{p,q}}, U)$$
$$= \mathrm{Ord}_v(T_{p,q} \# \overline{T_{p,q}})$$
$$= p-1 \quad (0 < p < q)$$

So d_r can be arbitrarily large among slice (ribbon!) knots. Currently, Sarkar's lower bound on d_r from Khovanov homology cannot prove this, as we (by whom I mean Manolescu - Marengon) have examples of K with that torsion order 3, but none greater.

Strongly homotopy-ribbon concordances (with Ian Zemke)

$(S^3 \times I) \setminus \nu(\Sigma) = (S^3 \setminus \nu(K)) \times I$

$\cup$ 1- and 2-handles



Thm (MZ)

If Σ strongly htpy-ribbon then

$$F_{\Sigma} : \widehat{HFK}(K) \hookrightarrow \widehat{HFK}(J)$$

(vertical decoration)

Pf. Same idea. Consider $\Sigma \cup \overline{\Sigma}$, decorated so w region is vertical strip.

$$(S^3 \times I) \setminus \nu(\Sigma \cup \overline{\Sigma})$$

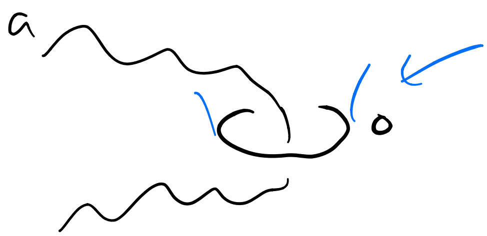
$$= (S^3 \setminus \nu(K)) \times I$$

$\cup$ 1-handles + 2-handles

upside down upside down

$\cup$ 2-handles + 3-handles

$\rightsquigarrow$ get Kirby diagram (in $S^3 \setminus \nu(K)$) where 2-handles in pairs like



3-handles only meet these types of 2-handles ("meridians")

Tube 2-spheres to K to achieve crossing changes with non-meridian 2-handles

(tube on cocore of non-meridian + core of meridian)

By Zemke, this doesn't chang $\mathcal{F}_{\Sigma \cup \bar{\Sigma}}$. Note 3-handle attaching spheres lie close to 2-handle attaching circles, so eventually $K \subset B^3$, B^3 disjoint from all attaching spheres.

Then $\mathcal{F}_{\Sigma \cup \bar{\Sigma}}$ factors as

$$\left(\mathcal{F}_{K \times I \subset B^3 \times I}\right) \# \left(\mathcal{F}_{\emptyset \subset (S^3 \setminus B^3) \times I}\right)$$

$$= \mathrm{id}_{\widehat{HFK}(K)}.$$

$\blacksquare$