Relative Trisections (just a reminder)

[Gay-Kirby 2012]

X^4 = compact, connected, orientable 4-mfd with nonempty boundary

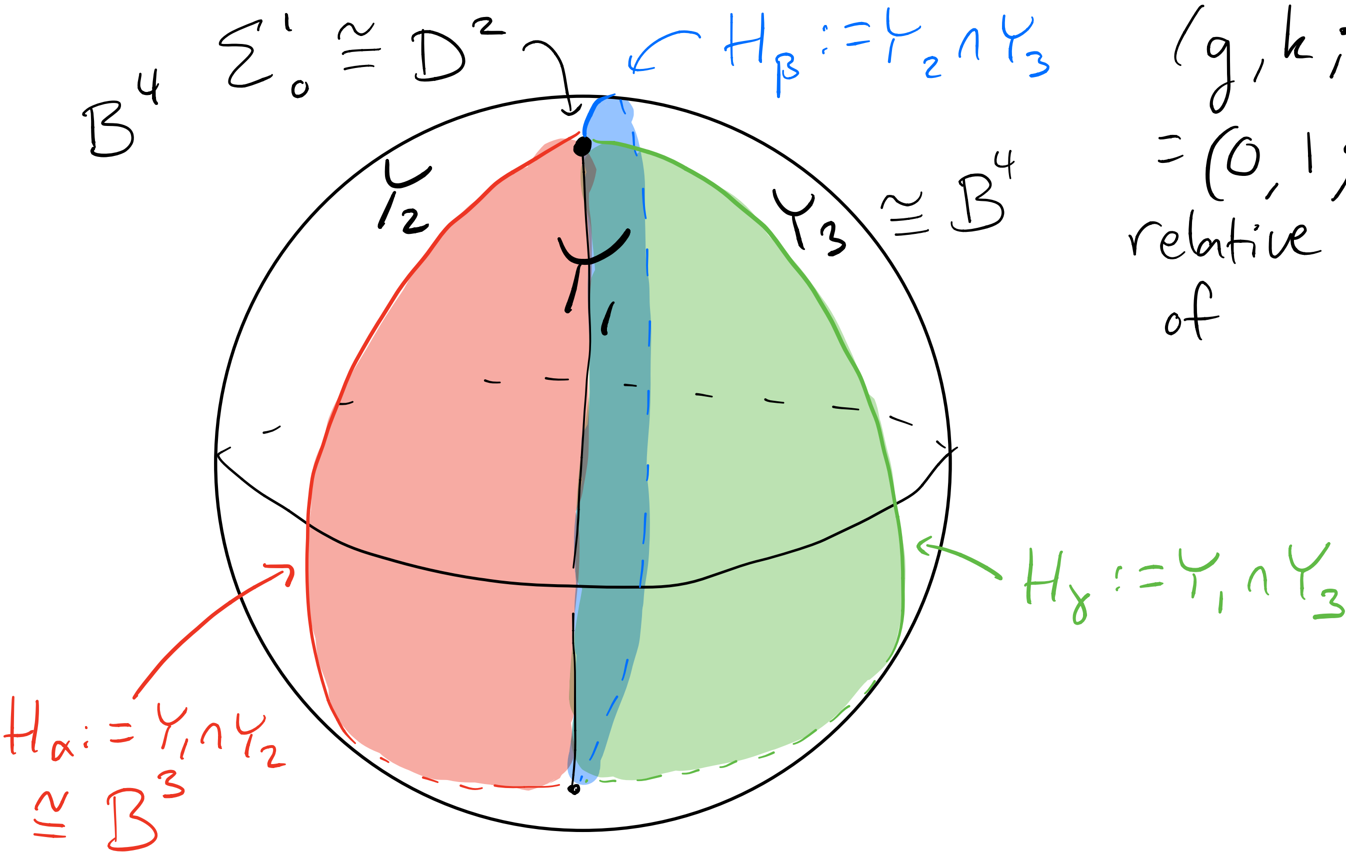
A $(g, \underbrace{k}_{k_1,k_2,k_3}; p, b)$ relative trisection of X is a decomposition $X = Y_1 \cup Y_2 \cup Y_3$ where

Analogous to trisection of closed manifold

- $Y_i \cong \natural_{k_i} S^1 \times B^3$ } 4D pieces are handlebodies
- Y_i, Y_{i+1} intersect only in their boundaries, $Y_i \cap Y_{i+1} \cong \natural_{p+b+g-1} S^1 \times D^2$ } double intersections are 3D handlebodies
- $Y_1 \cap Y_2 \cap Y_3 \cong$ genus-g surface with b boundary components Σ_g^b

open book on ∂X^4

- $Y_i \cap \partial X^4$ is a product
  $Y_i \cap \partial X^4 = (Y_i \cap Y_{i-1}) \times I$
  $(\Rightarrow = (Y_i \cap Y_{i+1}) \times I$ with opposite orientation$)$

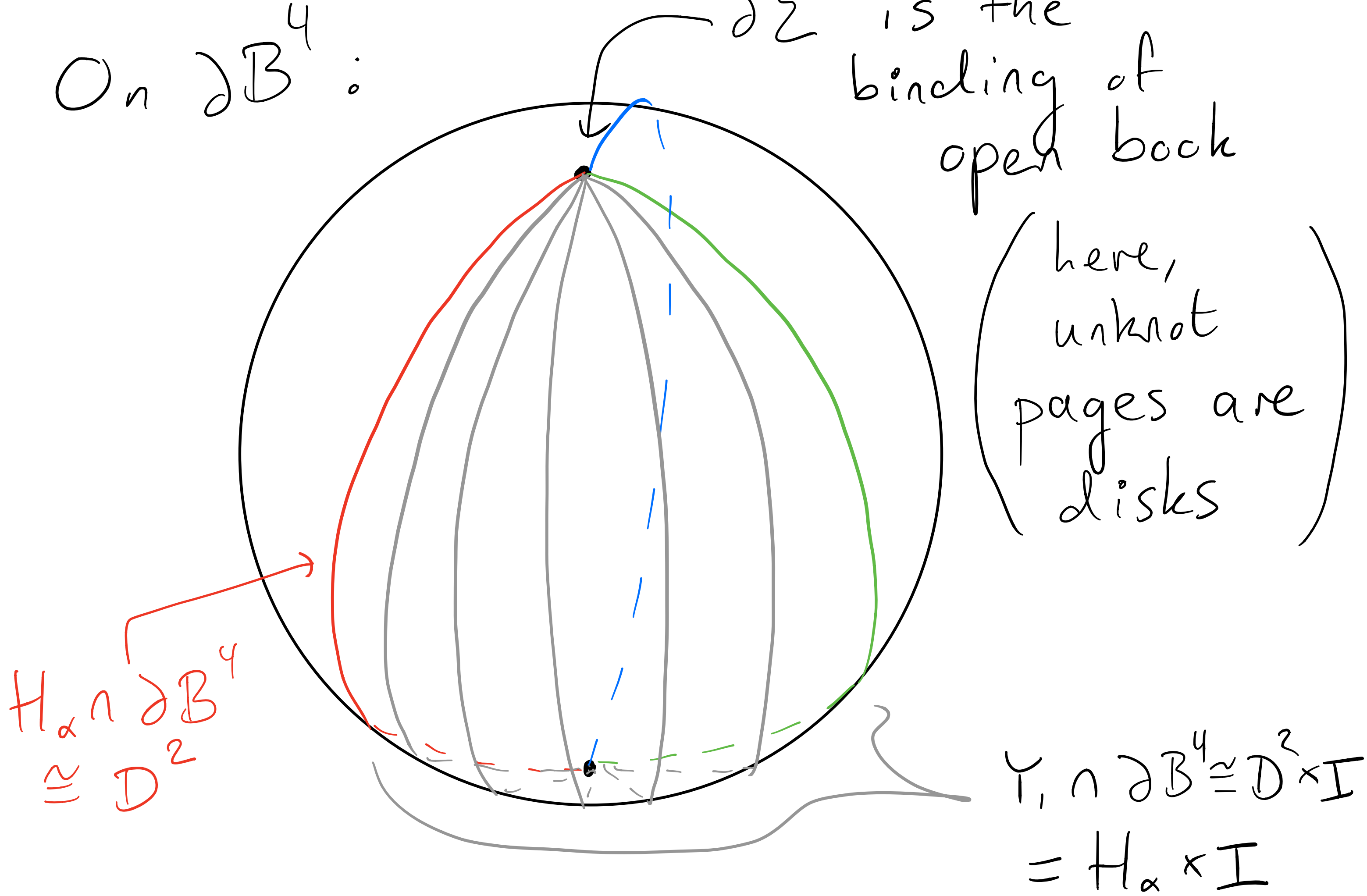


$(g,k;p,b)$
$= (0,1;0,1)$-relative trisection of B^4

On ∂B^4:

$\partial\Sigma$ is the binding of open book

(here, unknot pages are disks)



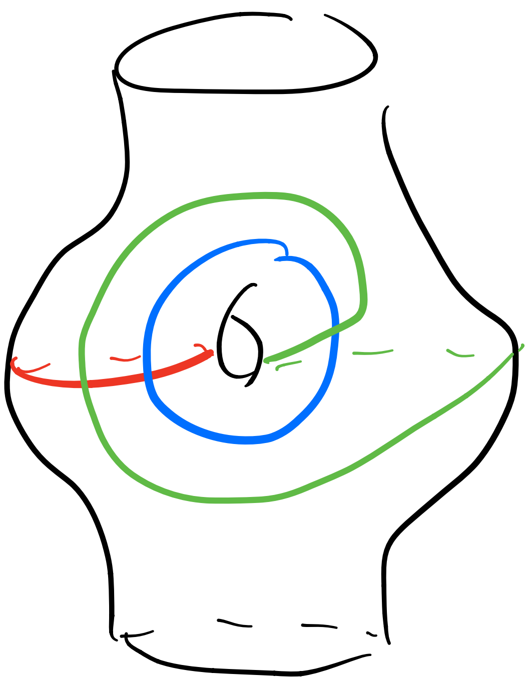
Note $H_\alpha \cap \partial X^4$ is a page of the open book.

(So are $H_\beta \cap \partial X^4$ and $H_\gamma \cap \partial X^4$).

We call $H_\alpha \cap \partial X^4$ the <u>α-page</u>

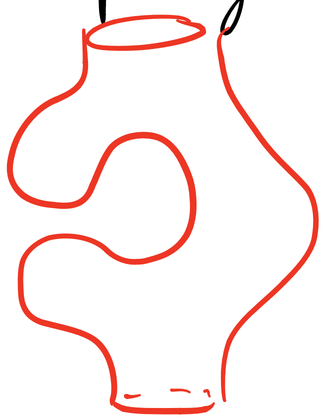
($H_\beta \cap \partial X^4 = \beta$-page, $H_\gamma \cap \partial X^4 = \gamma$-page.)

The α-page is parallel to Σ compressed along α.

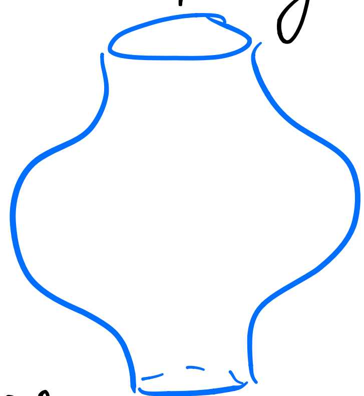


$(\Sigma, \alpha, \beta, \gamma)$

α-page



β-page



γ-page harder to draw

Point of the open book criteria:
So that diagrams still define a manifold.

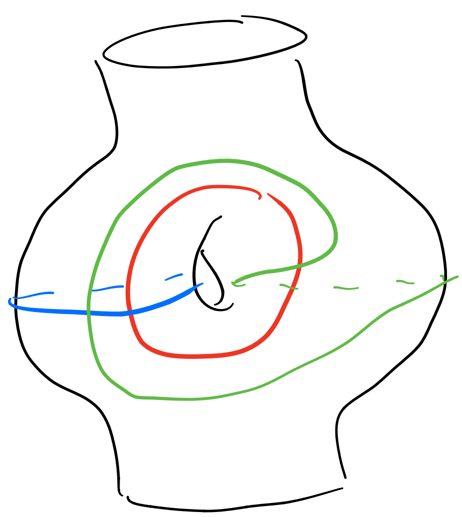
Diagram for relative trisection $X = Y_1 \cup Y_2 \cup Y_3$
$(\Sigma_g^b, \alpha, \beta, \gamma)$ where

$H_\alpha := Y_1 \cap Y_2$ is obtained from $\Sigma_g^b \times I$ by attaching 3D 2-handles to $\alpha \times 1$

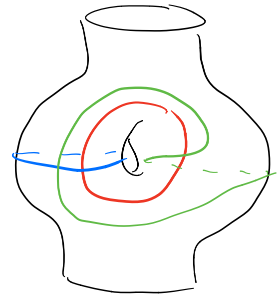
$H_\beta :=$ " $\beta \times 1$

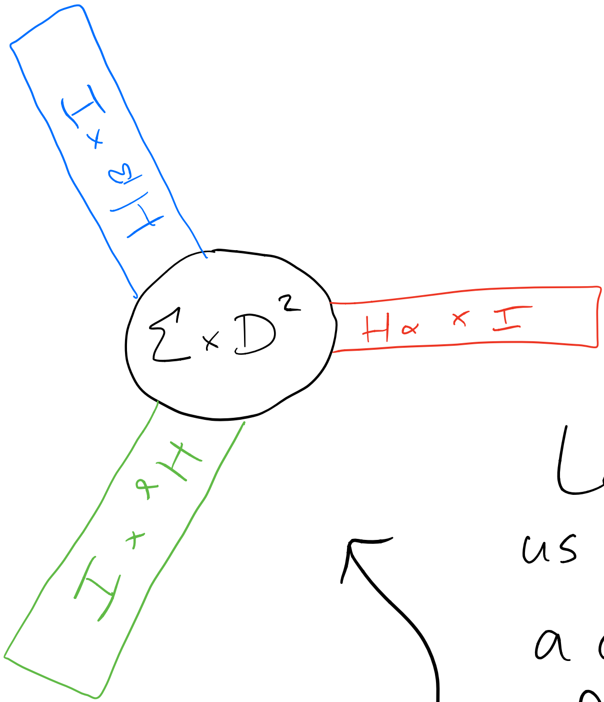
$H_\gamma :=$ " $\gamma \times 1$

e.g.

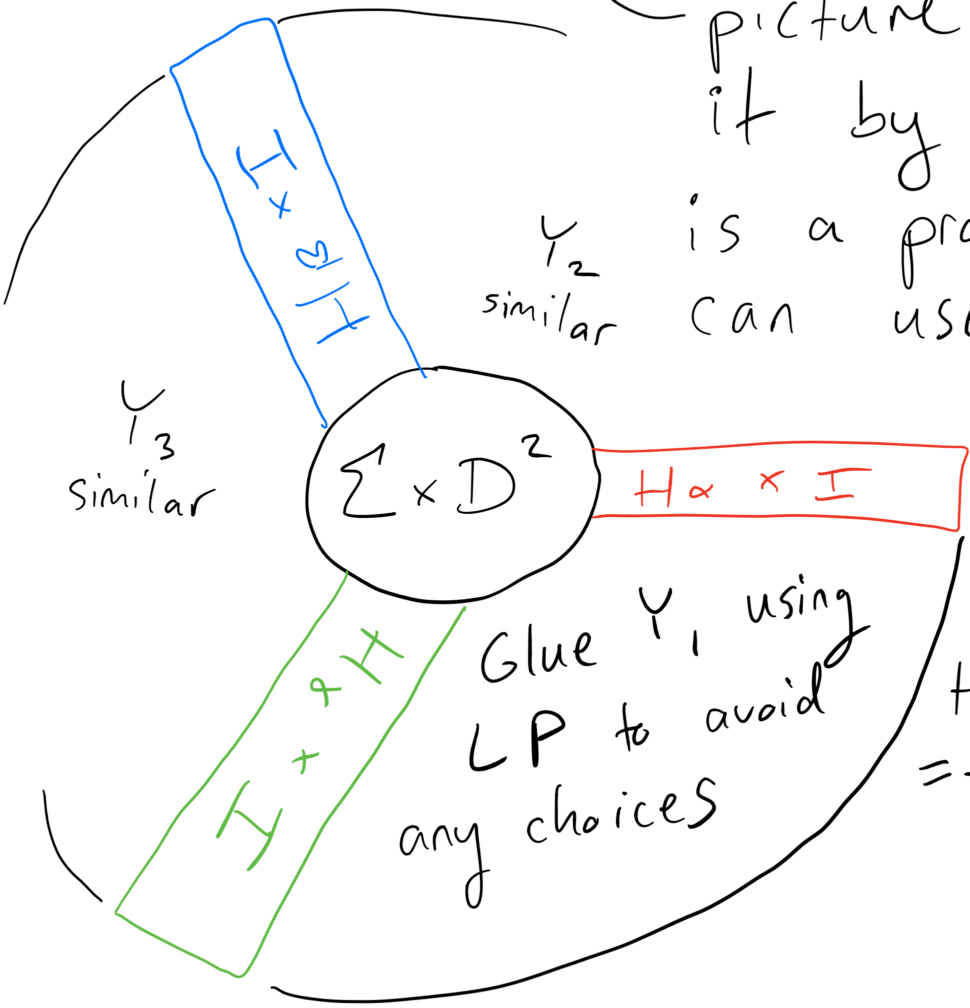


To recover X^4 from $(\Sigma, \alpha, \beta, \gamma)$:





Now just need to glue in Y_1, Y_2, Y_3. In closed case, Landenbach-Poenaru lets us do this without making a choice. Now, some of ∂Y_i isn't in this picture, so we specify it by saying $Y_i \cap \partial X^4$ is a product. Now we can use LP.



Y_2 similar

Y_3 similar

Glue Y_1 using LP to avoid any choices

$$H_\gamma \cap \partial X^4 \times I = -H_\alpha \cap \partial X^4 \times I$$

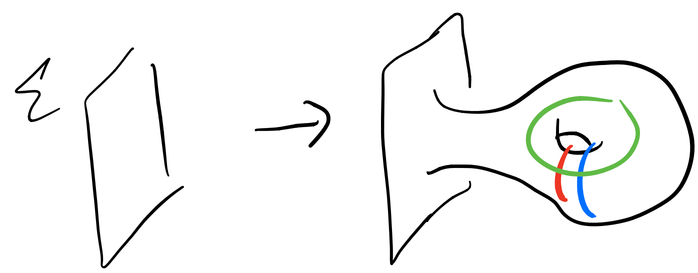
Theorem (Gay-Kirby '12
Castro-Gay-Pinzón-Caicedo '16)

Every (cpct connected orientable $\partial \neq \emptyset$) X^4 admits a relative trisection.

Theorem (Gay-Kirby '12
Castro '16)

For any open book $\mathcal{O}$ on ∂X^4, there exists a relative trisection of X^4 inducing $\mathcal{O}$

$\rightsquigarrow$ So in general, relative trisections of X^4 are not related by interior stabilization



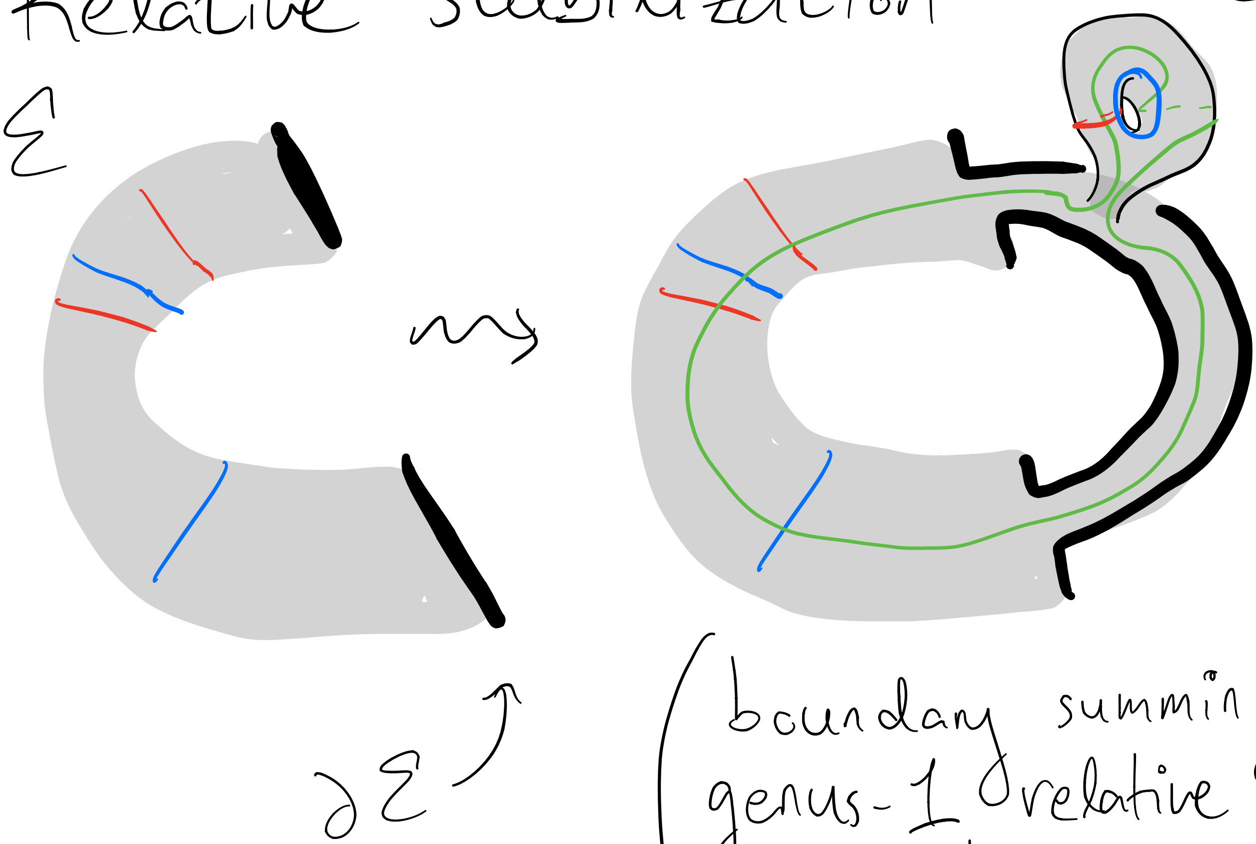
(connect-summing with a genus-1 trisection of S^4)

because this doesn't change the open book induced on ∂X.

Thm (Gay-Kirby 2012) If relative trisections T_1, T_2 of X^4 induce isotopic open books on ∂X^4, then they are related by interior stabilization.

Relative stabilization

Move introduced by Castro '16, studied diagramatically by Castro–Gay–Pinzón-Caicedo '16



(boundary summing with genus-1 relative trisection of B^4.)

Effect on open book: plumbing on Hopf band. (Sign depends on handedness of γ in the picture.)

Thm (Castro '16) (Consequence of above move + Giroux–Goodman)

If $\partial X^4 \cong \mathbb{Z}HS^3$, then any two relative trisections of X are related by a sequence of interior and relative stabilizations/destabilizations.

Key theorem (Piergallini-Zuddas 2018)

Any two open books of closed, compact, orientable, connected M^3 become isotopic after some number of

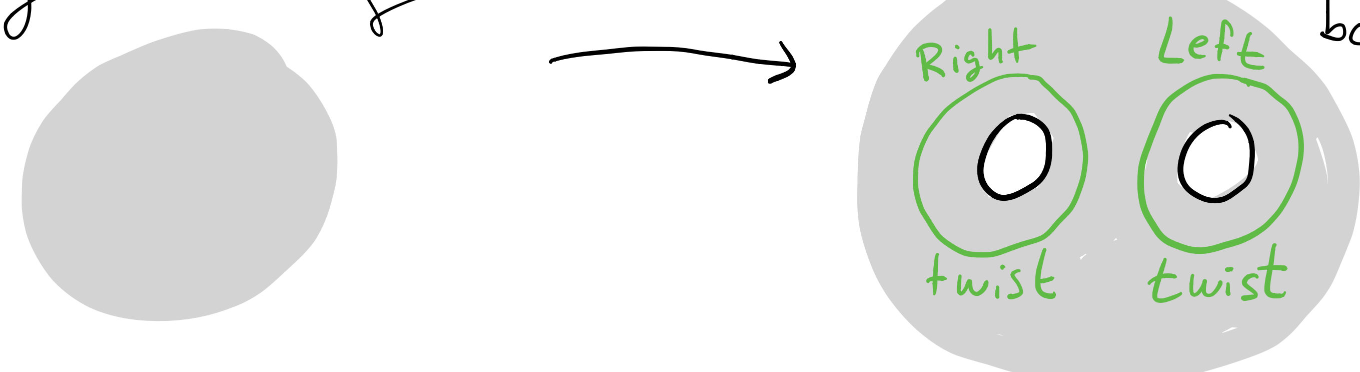
- Hopf stabilization / destabilization
- "∂U" and inverse

Open book

This disk in page is fixed pointwise by monodromy

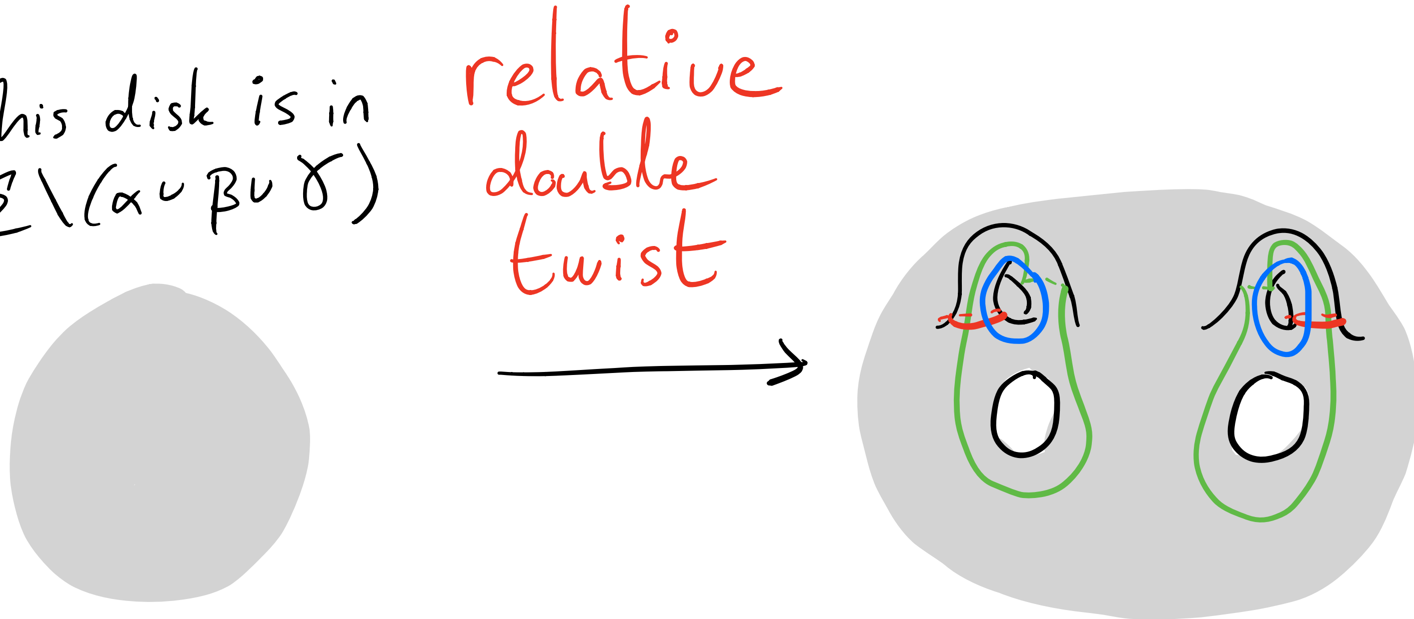
∂U

Puncture twice and add Dehn twists about new boundary to monodromy

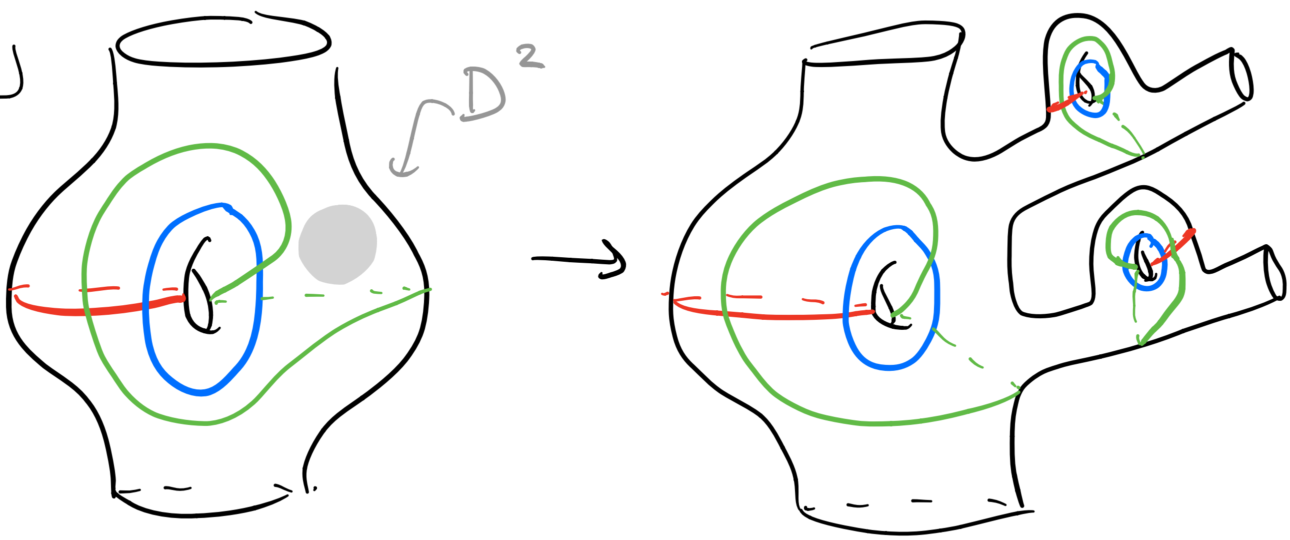


Corresponding trisection move
(Castro-Islambouli-M-Tomova)

This disk is in
$\Sigma \setminus (\alpha \cup \beta \cup \gamma)$

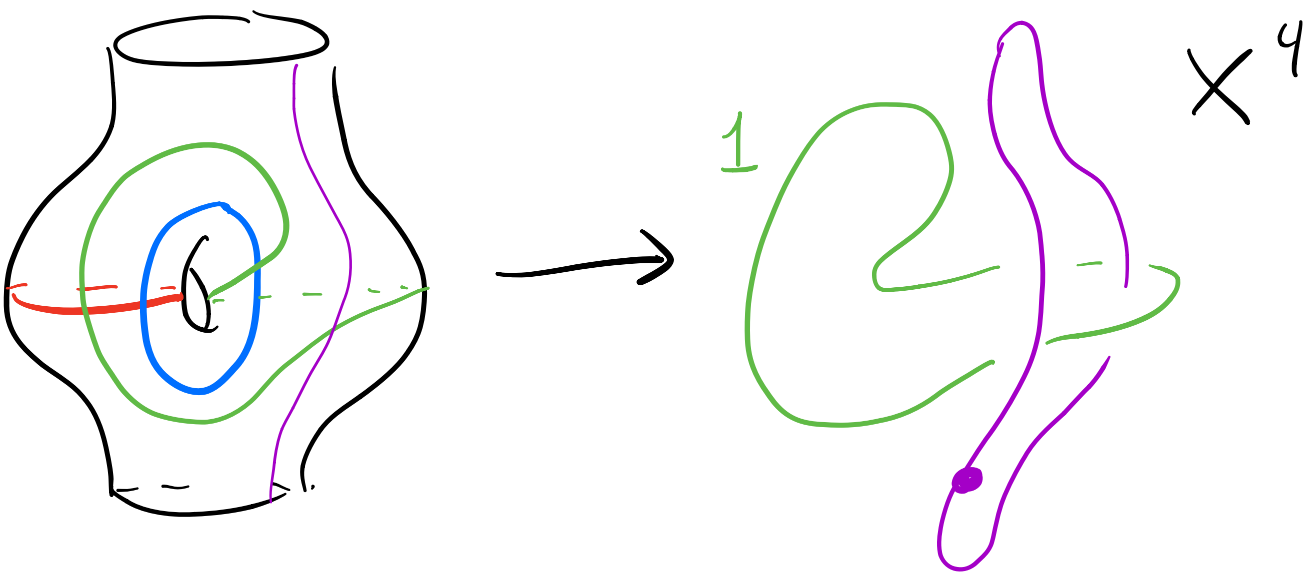


ex

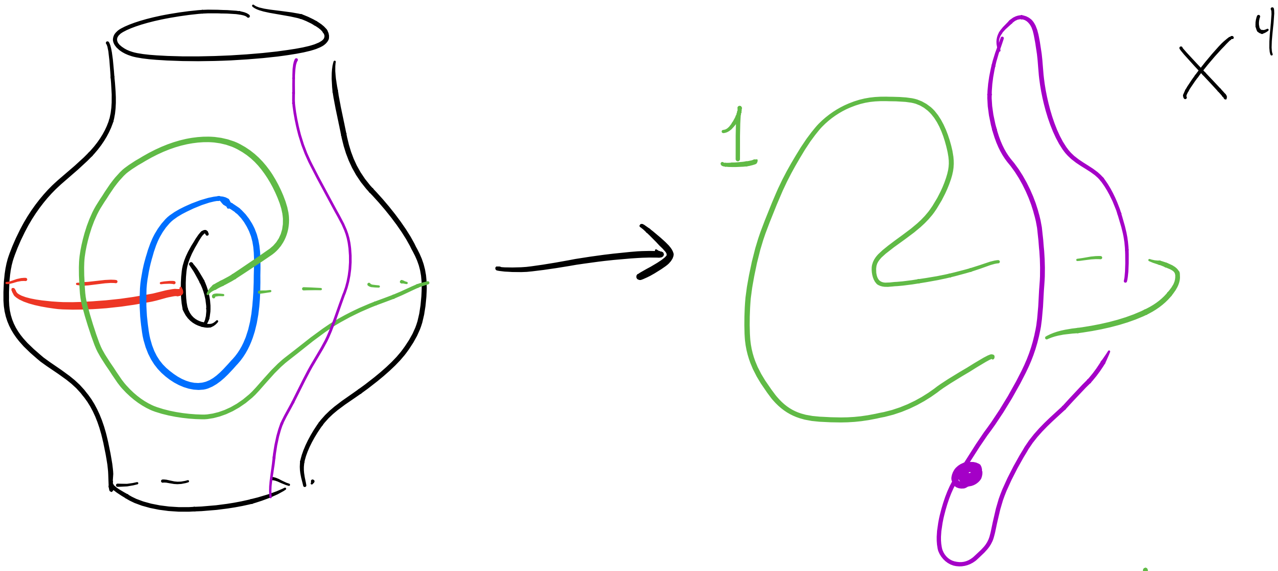


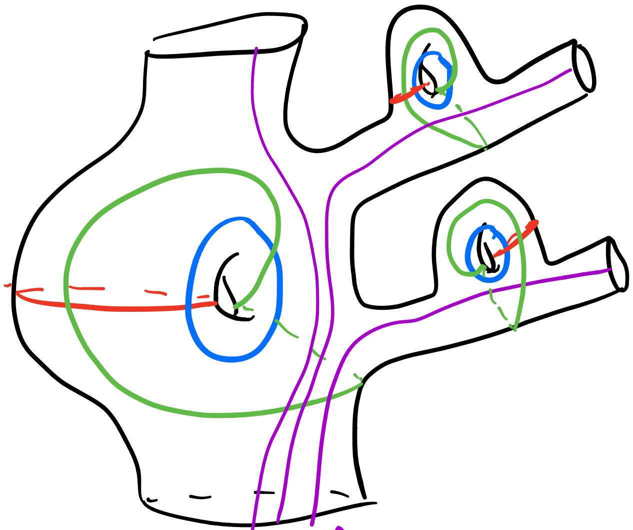
Check: What does this do to 4-mfd? (Kim-M 2018)

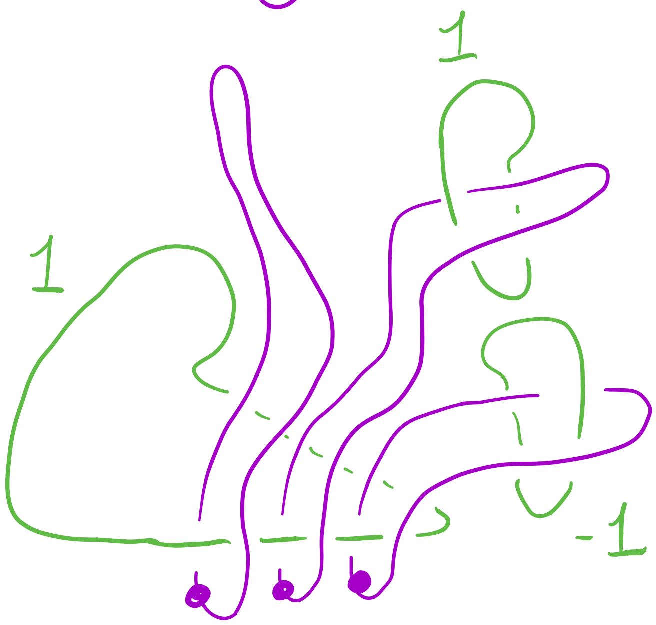
Recall that by standardizing (α, β), a relative trisection can yield a Kirby diagram.



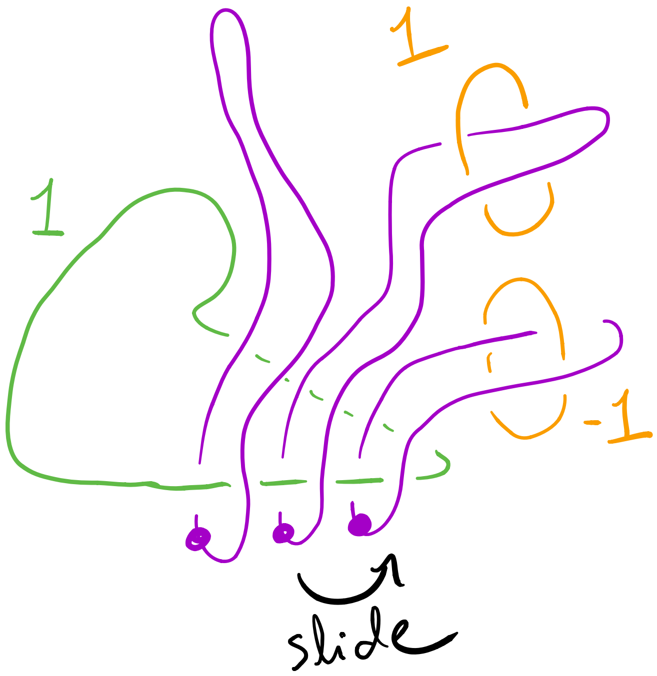
- parallel α, β and cut arcs for α,β page $\longrightarrow$ 1-handles
- γ dual to β $\longrightarrow$ 2-handles
- parallel α, γ $\longrightarrow$ 3-handles

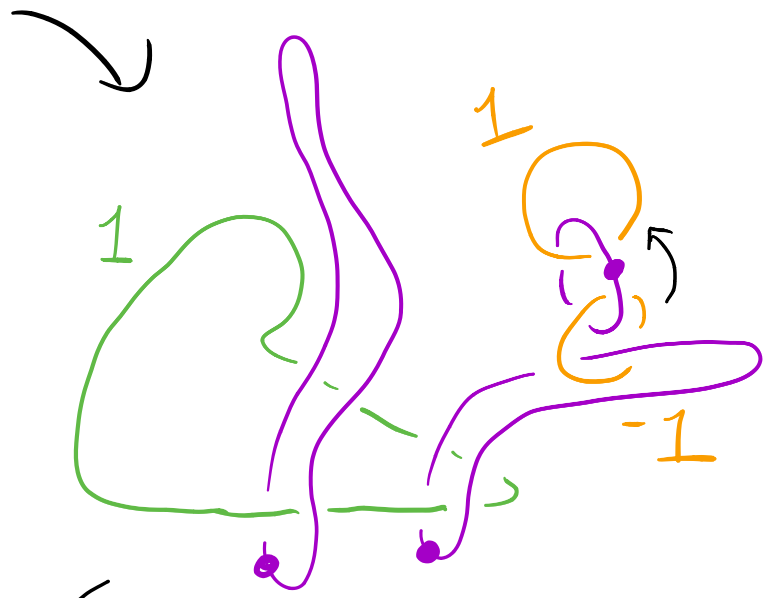


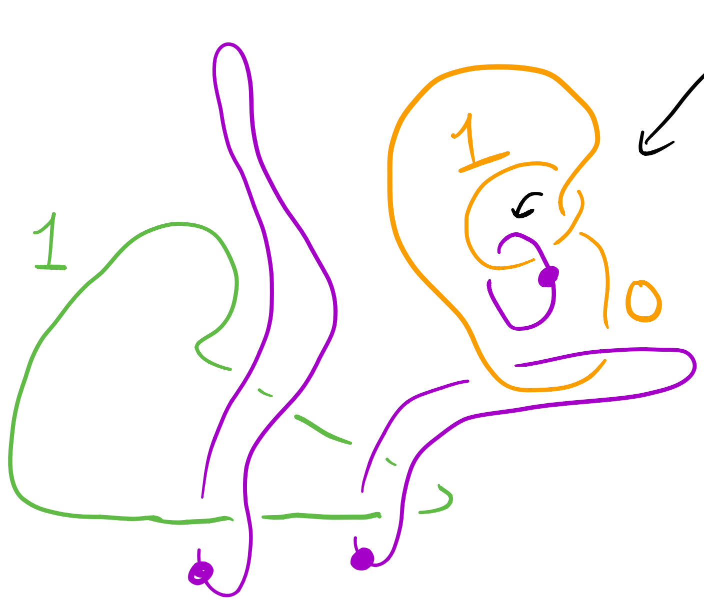


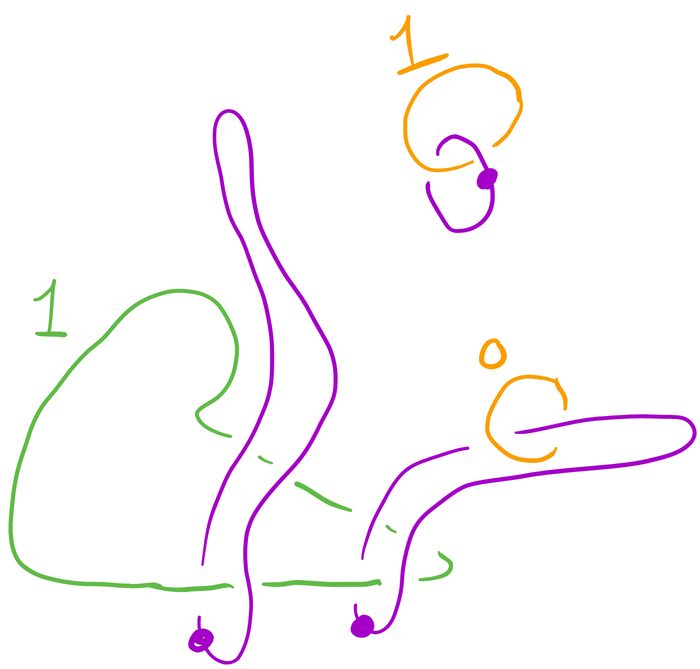


The new arcs are parallel away from the stabilization



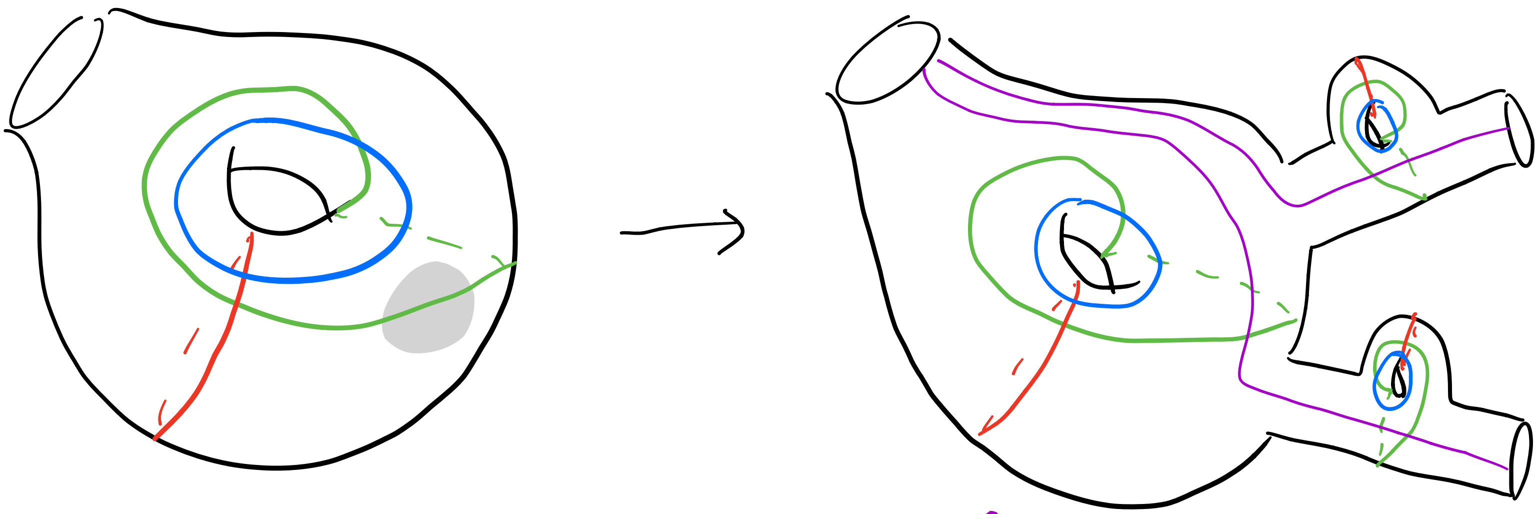






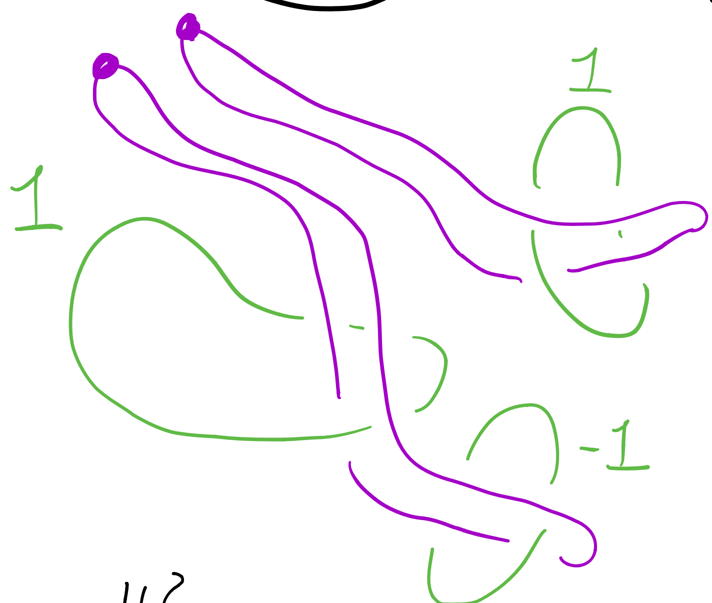
same 4-mfd
(so this is really a trisection move)

If we tried to relative double twist but allowed α, β, γ to separate the two punctures, the manifold could change.

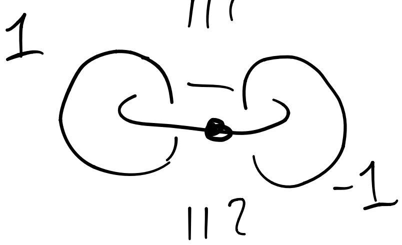


$\cong ?$

$\mathbb{CP}^2 \setminus \mathring{B}^4$

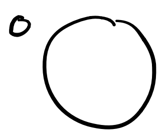


$\cong ?$



$= S^2 \times D^2$

$\cong ?$

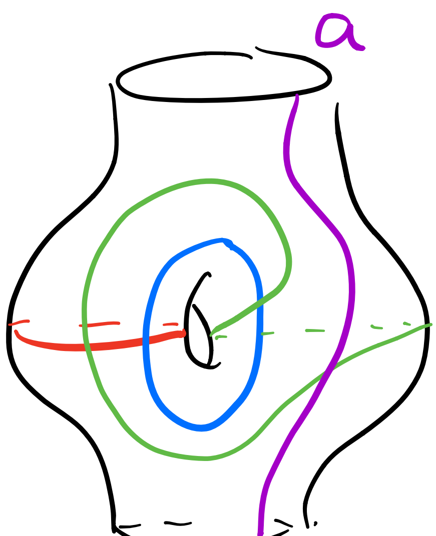


What does relative double twist do to boundary open book?

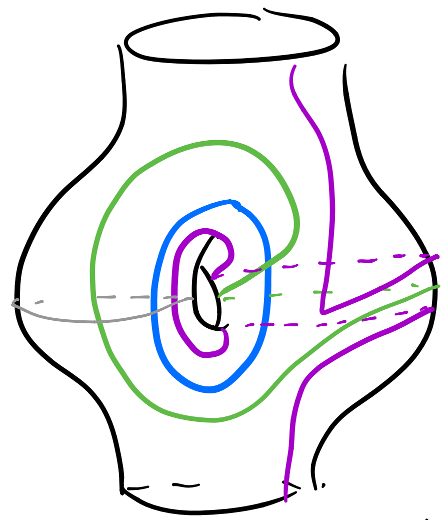
Recall: Monodromy of open book can be computed by algorithm (Castro–Gay–Pinzón-Caicedo '16)

(say α, β standard)

(Might also have to slide β curves)



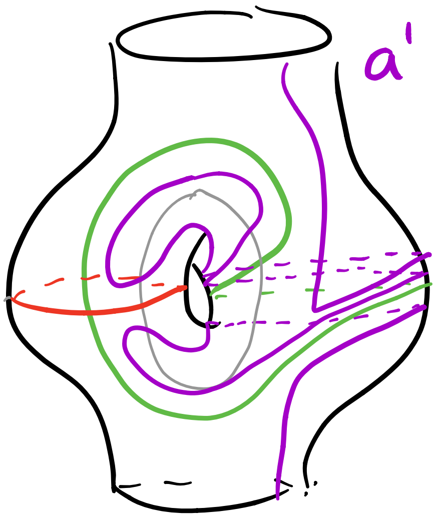
Step 1. Cut arcs a for α page.

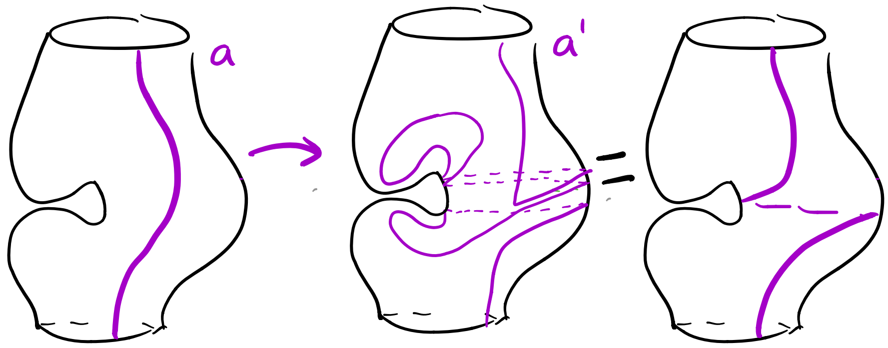


Step 2. Slide over β until disjoint from γ.

(Might also have to slide γ curves)

Page$(H \cup \partial X)$

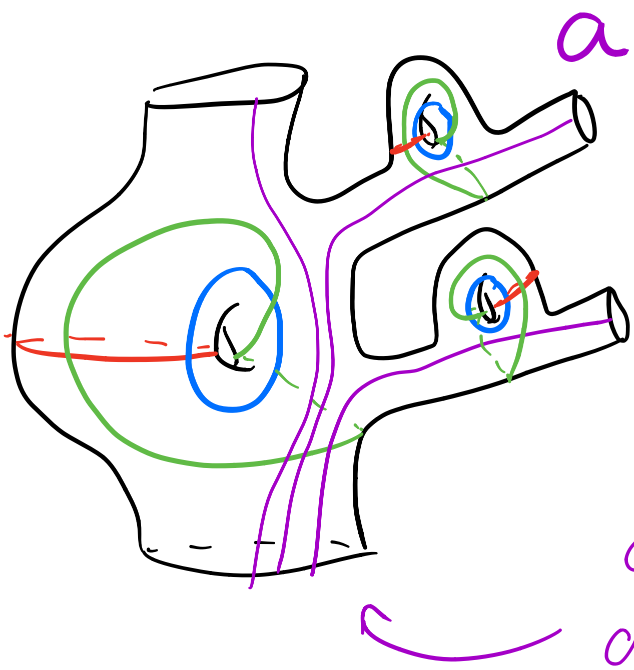




Step 3. Slide over γ until disjoint from α.

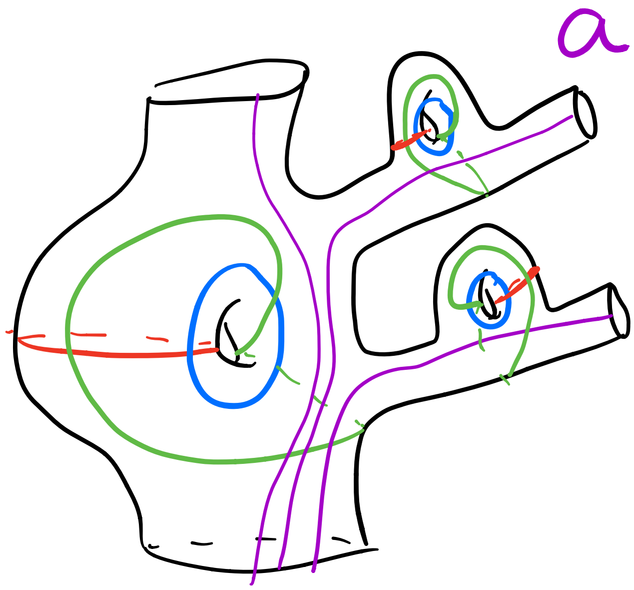
a' is the image of a under monodromy

After relative double twist,

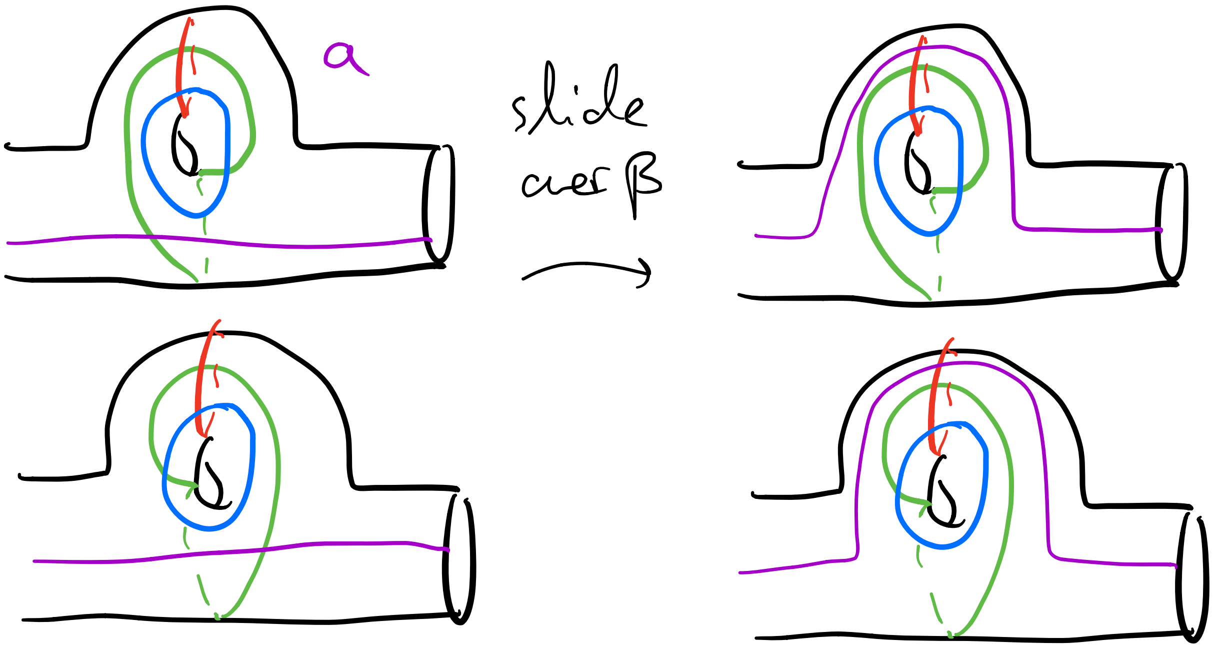


Can perform algorithm with same slides away from stabilization.

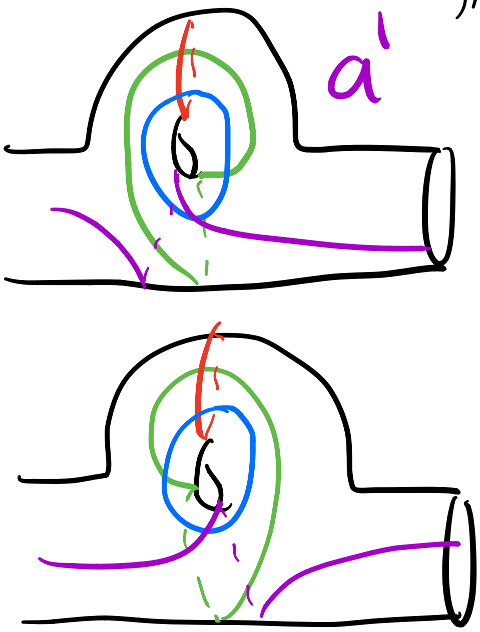
The new arcs are parallel away from the stabilization



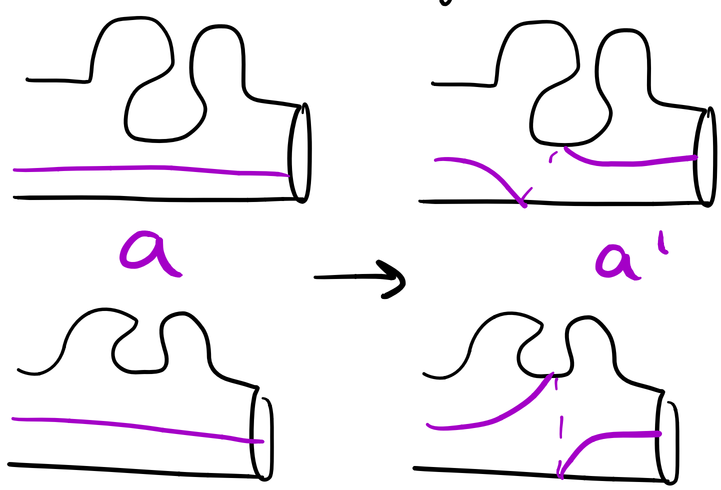
near stabilization, have:



slide over γ



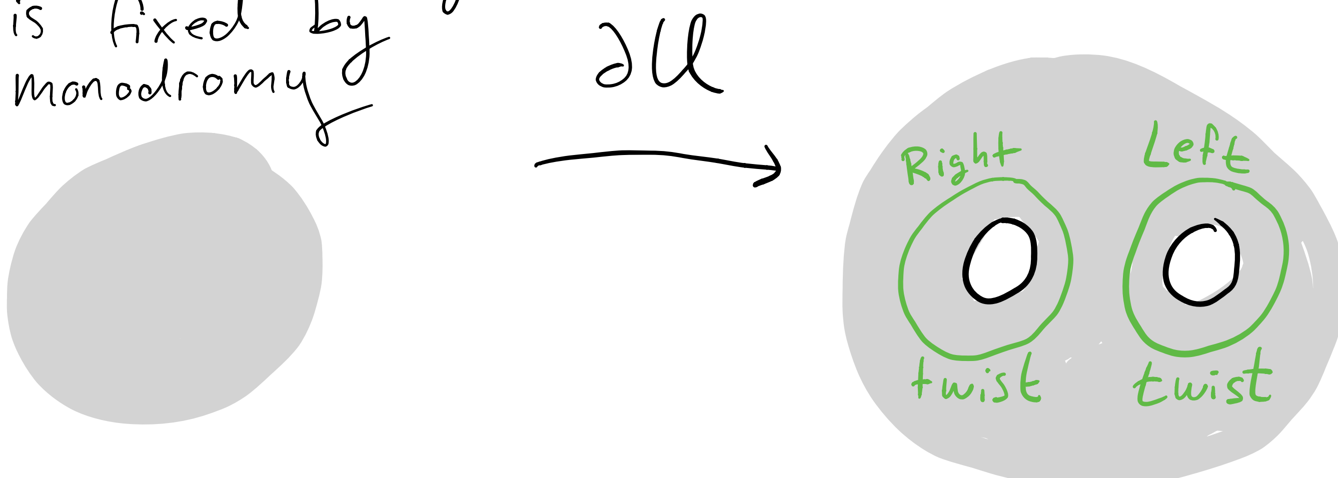
α page



$\therefore$ Effect of relative double twist on boundary open book is $\partial\mathcal{U}$!

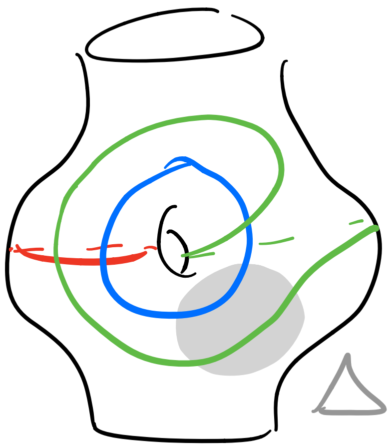
(We punctured the α page twice and changed the monodromy by adding opposite Dehn twists around the new boundaries)

This disk in page is fixed by monodromy



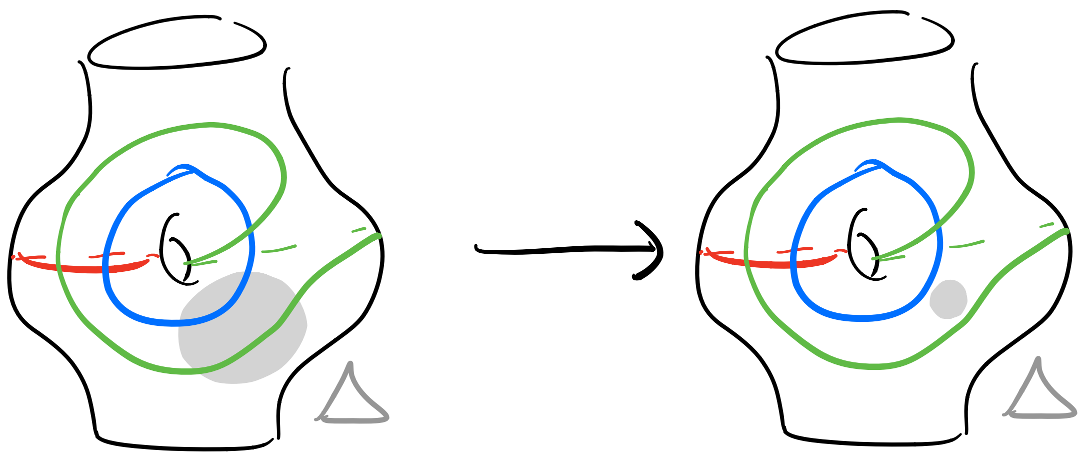
Moreover, any ∂U move can be achieved by a relative double twist.

Pf Let Δ be a disk in α-page fixed by monodromy.



In trisection diagram, $\Delta \longrightarrow$ disk disjoint from α curves.

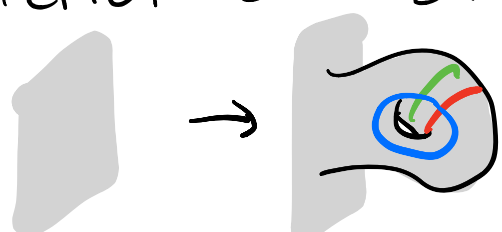
Since Δ fixed by monodromy, performing ∂U move at any two points (choices of punctures) in Δ achieves the same open book. So just shrink Δ to avoid β, γ curves and do relative double twist.



Consequence of (Castro-Islambouli-M-Tomova)

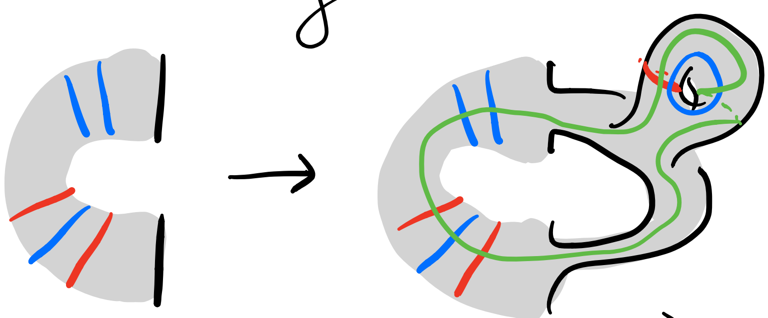
If T_1 and T_2 are relative trisections of a compact, connected, orientable 4-mfd X^4 ($\partial X^4 \neq \emptyset$) then T_1 and T_2 become isotopic after a finite number of

- interior stabilization



any permutation of red, green, blue

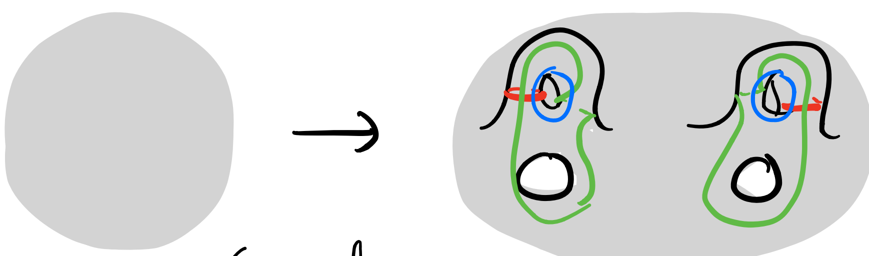
- boundary stabilization



Can be left or right twist

(and inverse)

- relative double twist



(and inverse)

Pf. By Piergallini–Zuddas, $\exists$ relative trisections T_1', T_2' s.t.

$$T_i \xrightarrow{\substack{\text{relative stabilizations,}\\ \text{double twists,}\\ \text{and inverses}}} T_i'$$

and T_1', T_2' induce isotopic open books on ∂X^4.

Then by Gay-Kirby, T_1' and T_2' have a common stabilization T.

$$\begin{tikzcd}
T_1 \arrow[r, "\partial \text{ moves}"] & T_1' \arrow[dr, "\substack{\text{interior}\\ \text{stabilization}}"] & \\
 & & T \\
T_2 \arrow[r, "\partial \text{ moves}"] & T_2' \arrow[ur, "\substack{\text{interior}\\ \text{stabilization}}"'] &
\end{tikzcd}$$