FINAL RESEARCH PROJECT

ABOUT THE EULER EQUATIONS:
THE PARTICLE-TRAJECTORY METHOD FOR
EXISTENCE AND UNIQUENESS OF SOLUTIONS

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Preface

The main goal of this project is to learn and accomplish the sufficient knowledge to understand the equations that govern an inviscid fluid in motion, with incompressible flow, also known as the Euler equations, as well as to study certain results of existence and uniqueness of local and global solutions in terms of the particle trajectories. Regarding to the first part, we mainly follow the notes from J. L. Vázquez [10], which give a clear and concise introduction to the fluid mechanics. In particular, we focus our special attention to chapters 1, 2 and 3, for the derivation of the Euler equations, chapter 5 to understand the properties of an ideal fluid and chapter 14 to study about the vorticity. We also follow chapters 1 and 2 from reference [6], by A. J. Majda and A. L. Bertozzi, which contain elementary material on incompressible flow, emphasizing the role of vorticity. The second part of this work is based in chapter 4 from the previous book, which studies the existence and uniqueness of solutions and the accumulation of vorticity for two and three dimensional flows.

The body of this project is divided in three chapters. First of all, we introduce in Chapter 1 the basic properties of fluids that satisfy the Euler equations. We may assume that the fluid behaves as a continuum medium so that all the quantities related, such as the velocity and the pressure, are continuous in time and space. We define the particle-trajectory map (see Definition 1.3) as the function that gives a particle’s position for a given time and a given initial position. We make in Section 1.3 an important distinction involving two different points of view: the Lagrangian reference, for which the main technique is to keep track of the location of individual particles and the Eulerian reference, where the velocity of the fluid plays an important role. We mainly use the Eulerian system because the equations are much simpler to handle.

Chapter 2 involves deeper physical results, from which we derive the Euler equations for an ideal fluid (see Definition 2.1). We apply two relevant physical laws in fluid dynamics. First, from the conservation of mass we obtain the continuity equation (2.3), which allows the particle trajectories to be volume preserving in time. On the other hand, Newton’s second law together with some of the fluid properties provide the conservation of momentum equation (2.5). In Section 2.4 we present the concept of vorticity, which is very important to understand the motion of ideal fluids. We also give the evolution vorticity equation (2.11) and we remark as an important fact that vorticity is conserved along particle trajectories for 2D flows.
and it is not preserved in the 3D case.

The final chapter is based on the study of existence and uniqueness of solutions. In Section 3.2 we see that the particle-trajectory equation is equivalent to the Euler equations and we show that they formulate an ODEs problem in terms of a linear operator for a certain Banach space. Then, Picard Theorem 3.3 provides the local existence of solutions. The tougher part in Chapter 3 is to prove that the mentioned operator satisfies the assumptions of Picard Theorem. This result is shown in Proposition 3.11 and in order to prove it, we have to present some technical lemmas, dealing with very useful potential theory estimates. As a final result, Beale, Kato and Majda (1984) give in Theorem 3.12 a sufficient condition, involving accumulation of vorticity, for which local solutions can be propagated globally in time. One of the main consequences from this result is that, since vorticity does not grow in time, there exist global solutions for 2D flows.

The Euler equations are a particular case of the well-known Navier-Stokes equations [2], which describe the motion of a *viscous* fluid. It is interesting to observe that there are currently open problems about these equations. For example, it has not been proved whether global solutions in three dimensional flows always exist.
Chapter 1
An Introduction to Fluid Mechanics

The study of fluid mechanics has a long history beginning from the ancient Greek times, where there was enough knowledge to solve certain flow problems. Archimedes (285-212 B.C.) in one of his books, *On Floating Bodies*, states the following principle:

Any object, wholly or partially immersed in a fluid, is buoyed up by a force equal to the weight of the fluid displaced by the object.

This statement was basically applied to floating and submerged bodies.

But it wasn’t until the 16th century that the first equations describing fluid flows were introduced. In 1502, Leonardo da Vinci (1452-1519) considered the conservation of mass with respect to the flow within a river, a one-dimensional steady flow. Moreover, in 1687, Isaac Newton (1642-1727) postulated his laws of motion and the law of viscosity of the linear fluids now called Newtonian. This theory was first based on the assumption that a fluid was perfect (see Section 1.4).

The era of fundamental discoveries took place during the 18th and 19th centuries, where very important theoretical results were proved. In 1750, Leonhard Euler (1707-1783) considered the conservation of momentum as Newton’s second law applied to fluids in motion with no viscosity. Euler developed both the differential equations of motion and their integrated form, now called the Bernoulli equation (see Chapter 2). In 1822, the equations that govern motion of a viscous fluid were first introduced by Navier (1785-1836), but a paper explaining accurately the derivation of the equations was not published until 1845 by Stokes (1819-1903). These are called the Navier-Stokes equations. In 1904, a German engineer, Ludwig Prandtl (1875-1953), published in 1961 [7], an important paper about fluid mechanics. Prandtl pointed out that fluid flows with small viscosity (water and air flows) can be divided into a thin viscous layer, near solid surfaces, patched on top of a nearly inviscid outer layer, where the Euler and Bernoulli equations apply.

In the 20th century, experimental methods have been used to measure flow velocities and fluid properties. In addition, due to the complexity of some fluid problems, numerical techniques have been developed in order to find approximate solutions.
1.1 Continuum assumption

Consider a fluid–gas or liquid–as a medium that deforms continuously when an arbitrarily small shearing stress acts on it. An example of a shear stress could be the friction due to fluid viscosity. As a consequence of the shearing force action, the fluid responses with a time-dependent deformation or movement (i.e., a fluid motion) called the flow.

Assuming that a fluid is described as a continuum means that every particle of the fluid can be indefinitely subdivided into smaller particles, and that each quantity involved tends to become constant over the smaller particles as their volumes tend to zero. Thus, all quantities of interest are assumed to be defined everywhere in space and to vary continuously from point to point within a flow. Furthermore, under this consideration, the variations in space and time can be accepted to be smooth and differential equations can be written to describe the fluid motion (see Chapter 2).

Note that a fluid is composed of multiple discrete molecules that move freely in a chaotic way, so there is no guarantee whatsoever that molecules are present at that point, at a given instant of time. In order for the continuum assumption to be valid, the length scale of the flow has to be much smaller than the experimental lengths, that we usually take as greater than $10^{-5}$ cm, in such a way that we only perceive an average of the individual processes between the particles. For example, suppose that we want to determine the density, mass per unit volume, at a point $C$. Let $\delta V$ be an infinitesimal volume of fluid and let $\delta V_0$ be the minimum volume that can be considered as a continuous and homogeneous medium. Then, we define the average density as

$$\rho = \lim_{\delta V \to \delta V_0} \frac{\delta m}{\delta V},$$

where $\delta m$ is the instantaneous number of molecules in $\delta V$. Since point $C$ was arbitrary, the density at any other point in the fluid could be similarly determined and hence, we can obtain an expression for the density distribution as a function of space and time.

So, the basic idea is that we take the limit value of the average quantities, as $\delta V$ tends to a smaller continuous medium. The same happens with other basic magnitudes such as velocity and pressure, force per unit of area applied uniformly over a surface.

1.2 Particle trajectories

Once we have established the continuum assumption, we describe mathematically how the fluid moves continuously as follows.

Let $\Omega_0 \subset \mathbb{R}^N$, $N = 2, 3$, be a bounded domain and $X_t : \Omega_0 \to \Omega_t$ be a family of transformations, where $\Omega_t$ are bounded domains in $\mathbb{R}^N$ and $t \in I = [0, T]$, with $T > 0$. We will make the following regularity hypotheses:

(i) $X : \Omega_0 \times [0, T] \to \mathbb{R}^N$ and $X(\alpha, t) = X_t(\alpha)$ is differentiable.
(ii) For every $t$ in $I$, $X_t$ is a diffeomorphism.

(iii) $X_0$ is the identity map.

The domain $Ω_0$ is the initial position, or space that occupies the fluid as a continuum medium, $Ω_t$ is the domain occupied by the continuum at a certain time $t$ and $X_t$ is the deformation map.

Let $x = (x_1, x_2, \ldots, x_N)$ be the position of the fluid’s particle at a time $t$ and $α = (α_1, α_2, \ldots, α_N)$ be the position of the same particle at time $t = 0$, which is called the reference position. We have that $α \in Ω_0$, $x \in Ω_t$ and the relation between them is

$$x = X(α, t).$$

**Definition 1.1.** For a fixed time $t$, we define the Jacobian matrix as

$$M := \left( \frac{∂x_i}{∂α_j} \right)_{i,j \in \{1,2,\ldots,N\}} = \nabla_α X_t,$$

also called the deformation matrix. The determinant of the transformation matrix is called the Jacobian determinant and we will denote it as

$$J(α, t) := \det(\nabla_α X_t), \text{ for } α \in Ω_0, \ t > 0.$$

Under the regularity hypothesis mentioned before, $X(α, t)$ is differentiable. By the chain rule, it holds that

$$\left(\nabla_α X_t^{-1}\right) \circ \left(\nabla_α X_t\right) = I,$$

which means that the inverse transformation from time $t$ to time 0 is given by the matrix

$$M^{-1} = \nabla_α X_t^{-1}.$$

Assumption (ii) says that $\nabla_α X_t$ is not singular, and hence $J(α, t) \neq 0$. Since $X_0$ is the identity map, $\det(\nabla_α X_0) = 1$. Then, by a continuity argument, we deduce that the determinant of the matrix is positive and therefore, these transformations preserve the orientation of the flow.

Suppose we want to study the trajectory of a singular particle of the fluid flow. The easiest way to do it is to fix a point $α \in Ω_0$, where the particle is initially, and then analyze the curve

$$x(α, t) = x(t) = X(α, t).$$

The velocity vector can be defined by the formula

$$\dot{u}(α, t) = \frac{dx}{dt} = \frac{∂}{∂t} X(α, t). \quad (1.1)$$

Similarly, the acceleration vector is given by

$$\ddot{a}(α, t) = \frac{d^2x}{dt^2} = \frac{∂^2}{∂t^2} X(α, t).$$
We have seen that computing the function $X(\alpha,t)$, we can get a full description of the fluid motion. Observe that the independent variables are the initial position $\alpha$, and the time $t$. This representation of the fluid flow was introduced by Lagrange, where the main technique was to keep track of the location of individual fluid particles. The Lagrangian representation is useful to give us an idea of fluid flows in experiments, but the observation and analysis of trajectories of fluid particles gets difficult whenever we have to deal with a large amount of them.

An alternative approach, provided by Euler, consists in the observation of the fluid velocity at fixed positions. That is,

$$u = u(x,t),$$

where now $u$ is a vector field, since it depends on the position $x \in \mathbb{R}^N$. Suppose, in the Eulerian reference, we know the analytic expression of the function $u$ in terms of $x$ and $t$. Then, it is easy to find the particle trajectories by integrating the Cauchy problem

$$\begin{cases}
\frac{dx}{dt} = u(x,t), \\
x(0) = \alpha,
\end{cases}$$

which is an initial value problem for the given ODE. We should now ask ourselves: *Is there any solution for the system? If so, is it unique?* The answer to these questions can be found in the following fundamental result from the Theory of Ordinary Differential Equations [1]:

**THEOREM 1.2 (Picard-Lindel"{o}f Theorem).** *Consider the initial value problem:

$$\begin{cases}
\dot{x} = f(t,x), \\
x(t_0) = x_0.
\end{cases} \quad (1.2)$$

Suppose $f : [t_0-\epsilon,t_0+\epsilon] \times \overline{B(x_0,\beta)} \to \mathbb{R}^N$ is continuous and bounded by $M$. Suppose, furthermore, that $f(t,\cdot)$ is Lipschitz continuous for every $t \in [t_0-b,t_0+b]$. Then, (1.2) has a unique solution defined on $[t_0-b,t_0+b]$, where $b = \min\{\epsilon, \beta/M\}$.  

Thus, if $u$ satisfies the hypothesis of Theorem 1.2 then, we can state that there is a unique local solution,

$$x = X(\alpha,t) = X_{\alpha}(t),$$

defined for all $t$ on the interval given by the theorem.

**DEFINITION 1.3.** *The vector field

$$X(\cdot,t) : \alpha \in \mathbb{R}^N \to X(\alpha,t) \in \mathbb{R}^N$$

is called the particle-trajectory map.*
This function has the following interpretation: an initial domain $\Omega_0 \subset \mathbb{R}^N$, in a fluid, evolves in time to $X(\Omega_0, t) = \{X(\alpha, t) : \alpha \in \Omega_0\}$, with the vector $u$ tangent to the particle trajectory (see Figure 1.1).

The use of both the Lagrangian and Eulerian formulations are important to understand all the properties of the fluids. Since we have different independent variables in each method, we have to pay attention to the different partial derivatives. It is clear that the transformation that gives us the change of variables between the coordinates is $x = X(\alpha, t)$. Let us define the function,

$$f(x, t) = f(X(\alpha, t), t) =: F(\alpha, t).$$

From now on, our usual notation will be Eulerian. Thus, for an arbitrary function $f = f(x, t)$ we will write

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial t} \bigg|_x, \quad \frac{\partial f}{\partial x_i} = \frac{\partial f}{\partial x_i} \bigg|_{t_x} (j \neq i) \text{ and } \nabla_x f = \nabla f.$$

On the other hand, for the Lagrangian derivatives, we find that

$$\frac{\partial F}{\partial t} \bigg|_\alpha = \frac{\partial f}{\partial t} \bigg|_x + \sum_{i=1}^N \frac{\partial f}{\partial x_i} \cdot \frac{\partial x_i}{\partial t} \bigg|_\alpha = \frac{\partial f}{\partial t} + \sum_{i=1}^N u_i \frac{\partial f}{\partial x_i}.$$

Finally, the fundamental relation we obtain to go from one reference to the other is:

$$\frac{\partial F}{\partial t} = \frac{\partial f}{\partial t} + u \cdot \nabla f. \quad (1.3)$$

### 1.3.1 Material derivative

Sometimes it may be confusing to use the double notation $f$ and $F$ to refer to the same quantity. To avoid that, we introduce the following notation for the Lagrangian derivative:

$$\frac{Df}{Dt} := \frac{\partial F}{\partial t}. \quad (1.4)$$
1.4. Classification of fluids

The material derivative, also called convective derivative, describes the time rate of change of some physical quantity subjected to a velocity field that depends on space and time. Looking at equations (1.3) and (1.4), we define the material derivative as

\[
\frac{D}{Dt} := \frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla.
\]

The second term of the above equation, \( \mathbf{u} \cdot \nabla \), is usually called the convective term and it plays an important role in studying processes that involve particles in motion. Taking rectangular coordinates, we see that

\[
\mathbf{u} \cdot \nabla f = \sum_{i=1}^{N} u_i \frac{\partial f}{\partial x_i}
\]

is the directional derivative of \( f \) along the vector \( \mathbf{u} \), which specifies the instantaneous rate of change of the function \( f \), moving at a velocity \( \mathbf{u} \).

1.4 Classification of fluids

The governing equations of motion for a fluid may vary, depending on the different properties of the flow. In this section, we will introduce the concept of a stress, which is fundamental to understand how components of a velocity field deform fluid particles. Very much related to this concept, we have the viscosity of the fluid, that measures the resistance to deformation, when acting a shear stress.

Fluids that obey Newton’s Law of Viscosity (1.5) are called Newtonian fluids. In particular, we will distinguish between viscid fluids and fluids with zero viscosity, also known as perfect fluids. In relation to Newtonian fluids, it is usual to study whether fluids have compressible or incompressible flows.

Therefore, in general terms, we will classify the fluids in four basic types: viscid-compressible, viscid-incompressible, perfect-compressible and perfect-incompressible. Our special interest will be in perfect and incompressible fluids, where the Euler equations apply (see Chapter 2).

1.4.1 Stresses

Fluid particles in a continuum medium can mainly experience two types of forces: surface or internal forces, such as pressure or friction, generated by contact with other particles, or external forces, such as gravity and electromagnetic.

Surface forces on a fluid particle lead to stresses. Quantitatively, a stress, in continuum mechanics, is a measure of the average force per unit area of a surface, on which internal forces act. In contrast with other quantities, like the density or the flow velocity, the surface stress depends not only on the space and time, but also on the orientation of the surface on which the stress acts, in general, given by the outward unit normal vector.

The force acting on the surface may be split into two components: one normal and the other tangent to the domain. Therefore, depending on whether we consider
the normal component of the force with respect to the surface, or the tangential component, we will talk about the normal stress $\sigma$ or the shear stress $\tau$, respectively. Let us now look more deeply into shear stresses.

### 1.4.2 Viscosity

Shear stresses within a flow arise due to viscosity, a physical property that characterizes the fluid resistance to deformation. Isaac Newton was the first to attempt a quantitative definition of the coefficient of viscosity. He considered a fluid element between two large parallel horizontal plates. The rectangular fluid element is initially at rest (see Figure 1.2). Suppose a constant rightward force $\delta F$ is applied to the upper plate, so that it moves along at a velocity $\delta u$, but the lower plate remains fixed. The angle of deformation of the fluid particle is $\delta \gamma$ (see Figure 1.3).

![Figure 1.2: Fluid element at rest.](image)

![Figure 1.3: Fluid element at time $\delta t$.](image)

The shearing action of the plates produces a shear stress $\tau_{yx}$, parallel to the fluid’s surface. If $\delta F_x$ is the force exerted by the plate on that element and $\delta A_y$ is the infinitesimal surface of the upper plate in contact with the fluid element, then the shear stress is defined as

$$\tau_{yx} = \lim_{\delta A_y \to 0} \frac{\delta F_x}{\delta A_y} = \frac{dF_x}{dA_y}.$$

Focusing on the time interval $[0, \delta t]$, when the fluid is subjected to shear stress, it experiences a rate of deformation $\dot{\gamma}$, called shear rate, which is given by

$$\dot{\gamma} = \lim_{\delta t \to 0} \frac{\delta \gamma}{\delta t} = \frac{du}{dy},$$

where $\delta y$ is the distance between the two plates. Newton postulated in his book, *Philosophie Principia Mathematica* (1687), that shear stress is directly proportional to shear rate:

$$\tau_{yx} = \mu \frac{du}{dy}. \tag{1.5}$$

The constant of proportionality $\mu$ is called viscosity coefficient. Equation (1.5) is known as Newton’s Law of Viscosity. Fluids that obey Newton’s Law of Viscosity
are called *Newtonian fluids*. For instance, water and air are Newtonian fluids. On the other hand, fluids in which shear stress is not directly proportional to shear rate are known as *non-Newtonian*.

In this monograph we will focus only on Newtonian fluids. The viscosity coefficient $\mu$ allows us to make an important distinction between two types of fluids: *viscid fluids* in which $\mu > 0$ and *perfect fluids*, where $\mu = 0$. Perfect fluids are of great interest in fluid mechanics since lack of viscosity in the equations describing this model are simpler to handle, as we will discuss in the next chapter. Before that, the next step will be to characterize a particular case of perfect fluids: fluids with incompressible flows.

### 1.4.3 Incompressible flows

In continuum mechanics, a flow is said to be incompressible if the density of a fluid particle does not change during its motion. Despite of the fact that all real fluids are compressible, we can assume in some cases that variations in density are negligible and hence, treat them as incompressible flows. For instance, mostly all flows of liquids may be considered incompressible due to the immense force required to change their density. On the other hand, the flow of gases is usually compressible.

Let us define an incompressible flow in a more formal way.

**Definition 1.4.** A flow $X(\cdot, t)$ is incompressible if the transformation map $X_t$ is volume preserving, for all positive time $t$. That is, given any subset in an initial domain $D_0 \subset \Omega_0 \subset \mathbb{R}^N$, with smooth boundaries, and any $t > 0$ it holds that

$$\text{Vol}(D_0) = \int_{D_0} dx = \int_{X_t(D_0)} dx = \text{Vol}(D_t),$$

where $X_t(D_0) = X(D_0, t) = D_t$.

Observe that by the change of variables formula, the previous condition may also be expressed as

$$J(\alpha, t) = \det(\nabla_\alpha X_t) = 1, \text{ for every } \alpha \in \Omega_0, \ t \geq 0.$$

Our goal now is to characterize incompressibility in terms of the velocity field of a fluid. But first, we need to state and prove some previous results for the particle-trajectory map or flow [6, 10].

**Lemma 1.5.** Let $\Omega_0 \subset \mathbb{R}^N$ be a bounded domain with a smooth boundary and let $X(\cdot, t)$ be a particle-trajectory map of a smooth velocity field $u \in \mathbb{R}^N$. Then

$$\frac{D J}{Dt}(\alpha, t) = (\nabla \cdot u)_{|X(\alpha, t), t} J(\alpha, t),$$

for all $\alpha \in \Omega_0$. 
Proof. By definition of $J$ and using Laplace expansion, for a fixed $i \in \{1, 2, \ldots, N\}$,

$$J = \det \left( \frac{\partial x_i}{\partial \alpha_j} \right)_{i,j} = \sum_j \frac{\partial x_i}{\partial \alpha_j} A_{ij},$$

where $A_{ij}$ is the minor of the element $\partial x_i/\partial \alpha_j$. Then, since the determinant is multilinear in columns, we compute the time derivative,

$$\frac{DJ}{Dt} = \sum_i \sum_j \frac{D}{Dt} \left( \frac{\partial x_i}{\partial \alpha_j} \right) A_{ij} = \sum_{i,j} \frac{\partial}{\partial \alpha_j} \left( \frac{Dx_i}{Dt} \right) A_{ij}.$$

According to (1.4) and Definition 1.1, we know that

$$\frac{Dx}{Dt} = \frac{\partial X}{\partial t} = u,$$

where $x = X(\alpha, t)$.

Hence, we get:

$$\frac{DJ}{Dt} = \sum_{i,j} \frac{\partial u_i}{\partial \alpha_j} A_{ij} = \sum_{i,j,k} \frac{\partial u_i}{\partial x_k} \frac{\partial x_k}{\partial \alpha_j} A_{ij}.$$

Minors satisfy the identity,

$$\sum_j \frac{\partial x_k}{\partial \alpha_j} A_{ij} = \delta_{ik} J, \quad \text{where} \quad d_{ik} = \begin{cases} 1, & k = i \\ 0, & k \neq i. \end{cases}$$

Therefore, we have that

$$\frac{DJ}{Dt} = \sum_{i,k} \frac{\partial u_i}{\partial x_k} \delta_{ik} J = \sum_i \frac{\partial u_i}{\partial x_i} J = (\nabla \cdot u) J.$$

The following formula, also called the Transport Formula, is very useful to determine the rate of change of a given function $f(x,t)$ in a certain domain $X(\Omega_0, t)$ moving with the fluid [6].

**Proposition 1.6 (The Transport Formula).** Let $\Omega_0 \subset \mathbb{R}^N$ be a bounded domain with a smooth boundary, and let $X(\cdot,t)$ be a given particle-trajectory map of a smooth velocity field $u$. Then, for any smooth function $f(x,t)$,

$$\frac{d}{dt} \int_{X(\Omega_0,t)} f \, dx = \int_{X(\Omega_0,t)} [f_t + \nabla(fu)] \, dx. \quad (1.6)$$

Proof. Using the change of variables $x = X(\alpha, t)$ we obtain:

$$\int_{X(\Omega_0,t)} f(x,t) \, dx = \int_{\Omega_0} f(X(\alpha,t),t)J(\alpha,t) \, d\alpha.$$
Since we are now integrating in a fixed domain (it does not depend on time), by the theorem of differentiation under the integral sign, and the relation given in (1.3), we have that
\[
\frac{d}{dt} \int_{X(\Omega_0,t)} f(x,t) \, dx = \int_{\Omega_0} \frac{\partial}{\partial t} \left[ f(X(\alpha,t),t)J(\alpha,t) \right] \, d\alpha \\
= \int_{\Omega_0} \left[ \left( \frac{\partial f}{\partial t} + u \cdot \nabla f \right) J + f \frac{\partial J}{\partial t} \right] \, d\alpha.
\]
Finally, using Lemma 1.5 and undoing the change of variables
\[
\frac{d}{dt} \int_{X(\Omega_0,t)} f(x,t) \, dx = \int_{\Omega_0} \left( \frac{\partial f}{\partial t} + u \cdot \nabla f + f \nabla \cdot u \right) J \, d\alpha = \int_{X(\Omega_0,t)} \left[ \frac{\partial f}{\partial t} + \nabla \cdot (fu) \right] \, dx.
\]
Under these conditions we can finally state the following theorem that describes an incompressible flow in terms of the velocity field (i.e., using Eulerian coordinates).

**Theorem 1.7.** A flow is incompressible if and only if
\[
\nabla \cdot u = 0.
\]

**Proof.** Take \( f \equiv 1 \) in the Transport Formula (1.6). Then we get,
\[
\frac{d}{dt} \int_{X(\Omega_0,t)} dx = \int_{X(\Omega_0,t)} (\nabla \cdot u) \, dx = \int_{\Omega_t} (\nabla \cdot u) \, dx.
\]
If a flow is incompressible, for every \( t \geq 0 \): \( \text{Vol}(\Omega_t) = \text{Vol}(\Omega_0) \). Thus,
\[
\int_{\Omega_t} (\nabla \cdot u) \, dx = \frac{d}{dt} \text{Vol}(\Omega_t) = \frac{d}{dt} \text{Vol}(\Omega_0) = 0.
\]
Since this is valid for every domain \( \Omega_t \), it holds that \( \nabla \cdot u = 0 \). Let us now assume that the divergence of the velocity field is zero. Again by the Transport Formula we have that
\[
\frac{d}{dt} \text{Vol}(\Omega_t) = 0, \text{ for every } t \geq 0,
\]
and hence, \( \text{Vol}(\Omega_t) \) is constant for every positive \( t \). In particular, \( \text{Vol}(\Omega_t) = \text{Vol}(\Omega_0) \), for all \( t \geq 0 \). \( \square \)
Chapter 2

Derivation of the Euler Equations

Euler (1755) was the first to provide a mathematical description of a perfect and incompressible fluid in motion. Euler applied Newton’s second law (2.4) to a fluid in motion under an internal force, known as the pressure gradient, to derive the now called Euler equations. Furthermore, these equations were the first partial differential equations that were ever written down.

The Euler equations were first published in Mémoires de l’Academie des Sciences de Berlin in 1757, in Euler’s article *Principes généraux du mouvement des fluides*. In this paper, the set of equations consisted in the conservation of mass and momentum equations. An additional equation, corresponding to the conservation of energy, was supplied by Pierre-Simon Laplace in 1816. Euler mentions several assumptions in his work, which we will now describe. The following paragraph has been extracted from an adapted translated version of Euler’s article [4]:

In order properly to understand the motion that will be imparted to the fluid it is necessary to determine, for each instant and for each point, both the velocity and the pressure of the fluid. [...] First of all, the nature of the fluid is assumed to be known, in which case it is necessary to consider its various forms since it may be compressible or incompressible. [...] It must also be assumed that the state of the fluid at a certain moment of time is known and I shall call this the initial state of the fluid. [...] Thirdly, the data must include the external forces to which the fluid is subjected. Thus, it could be assumed that the fluid is not exposed to any external force, unless it be natural gravity which is everywhere considered to be constant in magnitude and to act in the same direction.

Therefore, under these considerations, we will see in this chapter the derivation of the Euler equations by applying, in a control volume, some of the most relevant laws in physics, specially in the branch of dynamics, such as Newton’s second law and the conservation of mass. There are usually different ways to express these equations. The notation we will use along this work will be in the Eulerian system. But firstly, let us introduce the concept of an ideal fluid, since the governing equations of fluids motion, postulated by Euler, apply only for these type of fluids. We will also study some of their basic properties.
2.1 Ideal fluids

Let $\rho$ be the density of a fluid with respect to the position $x$ and time $t$. In order to work in a suitable frame, assume that $\rho(x,t)$ is a continuous and differentiable function. According to Definition 1.4 incompressible fluids preserve their volume over time. Moreover, mass will also remain constant with respect to time (as we will see in Section 2.2). Hence, it holds that

$$\frac{\partial \rho}{\partial t} = \frac{\partial}{\partial t} \left( \frac{m}{V} \right) = 0,$$

where $m$ and $V$ are the mass and the volume of the fluid, respectively. When this happens, the function $\rho$ is said to be stationary.

On the other hand, perfect and incompressible fluids are frequently homogeneous, which means that the density is constant with respect to space:

$$\nabla \rho = 0.$$

Thus, it is usual to make an additional hypothesis over the density: the homogeneous condition. Observe that, taking into account the two previous assumptions, the density will be constant throughout the whole domain.

**Definition 2.1.** An ideal fluid is a perfect, incompressible and homogeneous fluid.

Ideal fluids cannot really be found in nature, but they represent the simplest idealized model of a fluid in motion. Even though it is just an approximate model (actually that is what all models are) it gives us some general ideas and important information, that then can be applied to real situations. For instance, in many cases the viscous effects are concentrated near the solid boundaries while in regions far away from the boundaries the viscous effects can be neglected and hence, we can treat the fluid as it were ideal.

2.2 Conservation of mass

The principle of mass conservation states that the mass of a system must remain constant over time. Thus, mass can neither be created nor destroyed. In fluid dynamics, the term mass refers to the mass of the fluid, and the law implies that it is possible to measure the quantity of mass in an arbitrary domain and for a certain time, since it does not vary with the flow.

In mathematical notation, let $D_0 \subset \Omega_0$ be an initial bounded domain and consider the family $\{D_t = X_t(D_0)\}_{t \geq 0}$. For a fixed time $t$, $D_t$ is known as the control volume. Notice that this family determines how the system of fluid particles moves, considering the Lagrangian point of view (see Section 1.3). Then, it holds that for any positive time,

$$\text{Mass}(D_0) = \text{Mass}(D_t). \quad (2.1)$$
Let us now rewrite equation (2.1) in the integral form, for an arbitrary control volume. The principle of mass conservation can be written as

$$\int_{D_t} \rho(x, t) \, dx = \int_{D_0} \rho(\alpha, 0) \, d\alpha.$$ 

On the other hand, applying the change of reference from the Eulerian to the Lagrangian, \(x = X(\alpha, t)\), we obtain

$$\int_{D_t} \rho(x, t) \, dx = \int_{D_0} \rho(X(\alpha, t), t)J(\alpha, t) \, d\alpha,$$

where \(J(\alpha, t) = \det( DX_t(\alpha)) \). Therefore, it holds that for any domain \(D_0 \subset \Omega_0\),

$$\int_{D_0} \rho(\alpha, 0) \, d\alpha = \int_{D_0} \rho(X_t(\alpha), t)J(\alpha, t) \, d\alpha.$$

Since this equation is valid for an arbitrary initial domain, for all positive time \(t\) and for all \(\alpha\) in \(D_0\), we derive the Lagrangian version of the mass conservation,

$$\rho(\alpha, 0) = \rho(X_t(\alpha), t)J(\alpha, t). \tag{2.2}$$

In general, it is not very usual to work with the previous expression of the conservation law, since it becomes quite difficult to manage, when we apply it to each particle. Consequently, let us first consider the differential form of equation (2.2):

$$\frac{D}{Dt}(\rho J) = 0.$$

Observe that here, \(\rho\) and \(J\) are functions of \(\alpha\) and \(t\), and hence, the operator \(D/Dt\) corresponds to the material derivative seen in Section 1.3.1. By the chain rule,

$$0 = \frac{D}{Dt}(\rho J) = \frac{D\rho}{Dt} J + \rho \frac{D J}{Dt} = \left( \frac{\partial \rho}{\partial t} + u \nabla \rho \right) J + \rho \frac{D J}{Dt}.$$

We now apply the homogeneous hypothesis and Lemma 1.5. Then, since \(J \neq 0\), we obtain the Eulerian version of the mass conservation:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho u) = 0.$$

This equation is also referred as the continuity equation. Furthermore, as we stated in Section 2.1, \(\rho\) is not only homogeneous, but stationary. Thus,

$$\nabla \cdot u = 0 \tag{2.3}$$

To conclude, the scalar equation (2.3) is the principle of mass conservation in fluid dynamics, postulated by Euler, which is only satisfied by ideal fluids.
2.3 Conservation of momentum

In order to describe the equations governing a perfect fluid in motion, our next step is to derive the system of partial differential equations satisfied by the velocity field. Let us begin with Newton’s second law for a fluid’s particle:

\[
\frac{Dp}{Dt} = D\frac{D}{Dt}(mu) = ma = F. \tag{2.4}
\]

The quantity \(p\) is a vector with the same direction as the velocity, called the linear momentum or simply momentum. This law states that the change of momentum in a body, with respect to an inertial reference frame, is equal to the sum of all forces acting on it. Notice that the statement is written for a single particle, in the Lagrangian system. However, we need a general expression for the whole fluid in motion. That is, Newton’s second law defined in a control volume, with Eulerian coordinates. Let \(I(D_t)\) be the impulse (change in linear momentum) in an arbitrary domain \(D_t \subset \Omega_t\), for a fixed time \(t\). Then, equation (2.4) becomes

\[
\frac{d}{dt}I(D_t) = F(D_t),
\]

where \(F\) is the total force acting on the control volume \(D_t\). Moreover, in fluid dynamics, the total force is the sum of external forces, \(F_{\text{ext}}\), such as gravity, and surface forces, \(F_{\text{surf}}\), such as pressure. Since we assumed that external forces are negligible, the conservation of momentum in a certain domain \(D_t\) is given by the expression

\[
\frac{d}{dt}I(D_t) = F_{\text{surf}}(D_t).
\]

Let us now use some physical concepts and results to quantify, separately, each term of the equation. By definition of impulse,

\[
I(D_t) = \int_{D_t} u\rho \, dV.
\]

On the other hand, according to Augustin-Louis Cauchy (1822) [9], there exists a contact force density or Cauchy traction field, \(T(x,t,n)\), such that

\[
F_{\text{surf}}(D_t) = \int_{\partial D_t} T(x,t,n) \, dS,
\]

where \(n\) is the outward unit normal vector to the surface \(\partial D_t\). Notice that since ideal fluids are inviscid (i.e., \(\mu = 0\)), from Newton’s Law of Viscosity (1.5), it holds that there are no shear stresses. Thus, surface forces act only in the normal direction.

The following result was proved by Cauchy in 1827:

**Theorem 2.2 (Cauchy’s Theorem).** The contact force density \(T\) is a linear function with respect the normal vector \(n\). That is, there exists a function \(\sigma(x,t)\) such that

\[
T(x,t,n) = \sigma \cdot n,
\]

for all \(x \in \mathbb{R}^N\) and \(t \in \mathbb{R}\).
Definition 2.3. The function $\sigma$ satisfying Cauchy’s Theorem is called the stress tensor.

In coordinates, $\sigma \cdot n$ denotes the usual product between a matrix and a vector. There are many results dealing with the study of the stress tensor and all of its properties. In particular, we point out the following fundamental result for the derivation of the Euler equations [10]: for perfect fluids, it holds that

$$\sigma = -p(x,t)I_N,$$

where $p$ is the scalar internal pressure between the particles of the fluid and $I_N$ denotes the identity matrix of order $N$.

Another observation is that, regarding the impulse, we need to compute a volume integral. However, with respect to the internal forces, we have to calculate a surface integral. Since we are looking for an equation as simple as possible, we will use Gauss’s Theorem or the Divergence Theorem, in order to relate both quantities. Then, using the previous statements we conclude that

$$\frac{d}{dt} I(D_t) = \int_{D_t} \frac{d}{dt}(u\rho) \, dV = \int_{D_t} \rho \left[ \frac{\partial u}{\partial t} + (u \cdot \nabla) u \right] \, dV,$$

and

$$F_{\text{surf}}(D_t) = \int_{\partial D_t} -p \cdot n \, dS = -\int_{D_t} \nabla p \, dV.$$

Combining both right hand sides of the above equations, we have that

$$\int_{D_t} \left( \rho \left[ \frac{\partial u}{\partial t} + (u \cdot \nabla) u \right] + \nabla p \right) \, dV = 0.$$

Hence, since this equation holds for any domain $D_t \subset \Omega_t \subset \mathbb{R}^N$, we obtain Euler’s version of the conservation of momentum:

$$\frac{\partial u}{\partial t} + (u \cdot \nabla) u + \frac{1}{\rho} \nabla p = 0 \quad (2.5)$$

### 2.4 Vorticity evolution equations

A very important concept in fluid dynamics is the vorticity, which measures the tendency of a fluid to rotate. More precisely, the vorticity $\omega$ of a flow is defined as the curl of the the velocity field $u$ and it can be expressed by the formula:

$$\omega = \nabla \times u.$$

The vorticity is a key quantity in the analysis of the motion of incompressible and inviscid fluids since it establishes a significant difference between the solutions of the Euler equations in two and three dimensional flows, as we will see in the next chapter. Hence, it is convenient to study both cases separately.
2.4. Vorticity evolution equations

2.4.1 Two-dimensional flows

Consider a two dimensional flow with velocity field \( u = (u_1, u_2) \). The Euler conservation of momentum equations in this case are

\[
\begin{align*}
\frac{\partial u_1}{\partial t} + u \cdot \nabla u_1 + \frac{1}{\rho} \frac{\partial p}{\partial x_1} &= 0, \\
\frac{\partial u_2}{\partial t} + u \cdot \nabla u_2 + \frac{1}{\rho} \frac{\partial p}{\partial x_2} &= 0.
\end{align*}
\] (2.6)

The scalar curl of a 2D vector field \( u = (u_1, u_2) \) is the third component of the 3D curl of the vector field \( (u_1, u_2, 0) \). That is,

\[
\nabla \times u = \nabla \times (u_1, u_2) = \frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2},
\] (2.7)

We compute the derivatives of the first equation of (2.6) with respect to the second space variable \( x_2 \) and the second equation with respect to the first space variable \( x_1 \). Subtracting the resulting equations, we obtain:

\[
\frac{\partial}{\partial t} \left( \frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2} \right) + \frac{\partial}{\partial x_1} (u \cdot \nabla u_2) - \frac{\partial}{\partial x_2} (u \cdot \nabla u_1) = 0.
\]

Let \( \omega \) be the vorticity of \( u \). Then, using (2.7), it follows that the vorticity of a two-dimensional flow satisfies the transport equation [2]:

\[
\frac{\partial \omega}{\partial t} + u \cdot \nabla \omega = 0
\] (2.8)

2.4.2 Three-dimensional flows

For the 3D-case we will show an idea of the derivation of the vorticity evolution equation. First, we apply the linear operator curl to the Euler vector equation (2.5), to obtain the following expression:

\[
\frac{\partial \omega}{\partial t} + \nabla \times (u \cdot \nabla) u + \frac{1}{\rho} \nabla \times (\nabla p) = 0.
\] (2.9)

We present in the following lemma important vector calculus identities that will let us simplify (2.9).

**Lemma 2.4.** Let \( \varphi \) be a scalar field and \( F \) be a vector field. Then:

\( (i) \) \( \nabla \times (\nabla \varphi) = 0.\)

\( (ii) \) \( \nabla \cdot (\nabla \times F) = 0.\)

Since the pressure \( p \) is a scalar field, we have that \( \nabla \times (\nabla p) = 0. \) Regarding the middle term of the previous equation, we state that

\[
(u \cdot \nabla) u = \nabla \left( \frac{1}{2} u \cdot u \right) - u \times \omega.
\]
PROPOSITION 2.5. Let $u$ be a differentiable vector field in $\mathbb{R}^3$ and let $\omega = \nabla \times u$. Then:

$$\nabla \times (u \times \omega) = (\omega \cdot \nabla)u - (u \cdot \nabla)\omega.$$ 

Proof. We use another vector calculus identity, which holds for any differentiable vector fields:

$$\nabla \times (u \times \omega) = u(\nabla \cdot \omega) - \omega(\nabla \cdot u) + (\omega \cdot \nabla)u - (u \cdot \nabla)\omega.$$ 

Since our fluid is incompressible, we have that $\nabla \cdot u = 0$. Furthermore, applying Lemma 2.4, it holds that

$$\nabla \cdot w = \nabla \cdot (\nabla \times u) = 0.$$ 

Therefore, we have the result.

Finally, putting all these pieces together, we see that the vorticity of a three-dimensional flow satisfies the equation:

$$\frac{\partial \omega}{\partial t} + (u \cdot \nabla)\omega = (\omega \cdot \nabla)u \quad (2.10)$$ 

Observe that the vorticity evolution equations for 2D flows (2.8) and for 3D flows (2.10) are very similar, despite the fact that $(\omega \cdot \nabla)u = 0$ in the first case.

2.5 Vorticity-transport formula

It is interesting to consider the vorticity evolution equation (2.10) in the Lagrangian form, which is given by the expression,

$$\frac{D\omega}{Dt} = (\omega \cdot \nabla)u. \quad (2.11)$$ 

For 2D flows, given an initial position $\alpha$, vorticity remains constant over time because $D\omega/Dt = 0$, and for 3D flows, vorticity is not constant along the trajectories and furthermore, it varies significantly in magnitude. This result is given by the vorticity-transport formula, which proves that equation (2.11) can be integrated exactly by means of the particle-trajectory equation [6].

PROPOSITION 2.6 (Vorticity-Transport Formula). Let $X(\alpha, t)$ be the smooth particle trajectories corresponding to a divergence-free velocity field $u(x, t)$. Then the solution $\omega$ to the vorticity evolution equation (2.11) is

$$\omega(X(\alpha, t), t) = \nabla_\alpha X(\alpha, t)\omega_0(\alpha).$$ 

This formula gives a geometric interpretation of the stretching of vorticity, by means of the term $\nabla_\alpha X(\alpha, t)$. As a consequence, for two dimensions, we have the following result:
Corollary 2.7. Let $X(\alpha,t)$ be the smooth particle trajectories corresponding to a divergence-free velocity field. Then, the vorticity $\omega(x,t)$ satisfies

$$\omega(X(\alpha,t),t) = \omega_0(\alpha), \quad \alpha \in \mathbb{R}^2,$$

and the vorticity $\omega_0(\alpha)$ is conserved along particle trajectories for two-dimensional inviscid fluid flows.

To conclude, the vorticity is transported and stretched along particle trajectories for 3D flows and is conserved along particle paths for 2D flows. This leads to a natural reformulation of the Euler equations as an evolution equation for the vorticity alone, the vorticity-stream formulation, discussed in Section 3.1.
Chapter 3

The Particle-Trajectory Method

In Chapter 2 we derived the conservation of mass and the conservation of momentum evolution equations for the velocity field $u(x,t)$. Because the density $\rho(x,t)$ is homogeneous and stationary, we assume without loss of generality that $\rho \equiv 1$. Thus, the Euler equations with a given initial condition are

$$\begin{cases}
\frac{\partial u}{\partial t} + (u \cdot \nabla)u + \nabla p = 0, \\
\nabla \cdot u = 0, \\
u|_{t=0} = u_0(x).
\end{cases}$$

(3.1)

The main goal of this chapter is to prove the existence and uniqueness of solutions of the Euler equations in terms of the particle-trajectory maps, taking into advantage the good regularity properties. First, we introduce the vorticity-stream formulation of the Euler equations to show that the divergence-free velocity $u$ can be recovered from the vorticity $\omega$ by the convolution of the vorticity with a homogeneous kernel $K_N$. From here we derive the integro-differential equation for the particle trajectories. Proposition 3.2 shows that the particle-trajectory equations are equivalent to the Euler equations, for sufficiently smooth solutions with a rapidly decreasing vorticity.

The advantage of dealing with this new equations is that they define an ODEs problem in terms of a bounded linear operator on a certain Banach space. Thus, we can apply Picard Theorem 3.3 to prove local existence and uniqueness of solutions. But first, we have to see that this operator satisfies the required assumptions. For that, we define a Banach space and an open subset for which things work properly. We make as well a brief introduction to the Hölder spaces and we present some technical lemmas, the proof of which can be found in [6], that will give us useful potential theory estimates.

The final step is to show a sufficient condition, involving accumulation of vorticity, in order to propagate the local solution globally in time (see Theorem 3.12). This result is due to Beale, Kato and Majda and it gives a remarkable conclusion for 2D flows. We know from Corollary 2.7 that vorticity is conserved along particle trajectories so, as a consequence, there will be global-in-time existence and uniqueness of solutions in two dimensions.
3.1 The vorticity-stream formulation of the Euler equations

We have seen in Section 2.5 that taking the curl to the conservation of momentum Euler equation leads to the following evolution equation for the vorticity, $\omega = \nabla \times u$:

$$\frac{D\omega}{Dt} = (\omega \cdot \nabla)u.$$  

For 2D flows, $\omega$ is orthogonal to $u$ and hence, the vorticity stretching term vanishes: $(\omega \cdot \nabla)u = 0$. The vorticity-transport formula implies conservation of vorticity along particle trajectories $X(\alpha,t)$ (see Corollary 2.7):

$$\omega(X(\alpha,t),t) = \omega_0(\alpha), \quad \alpha \in \mathbb{R}^2.$$  \hspace{1cm} (3.2)

However, general 3D flows, have $(\omega \cdot \nabla)u \neq 0$ and the vorticity stretches according to the formula

$$\omega(X(\alpha,t),t) = \nabla_\alpha X(\alpha,t)\omega_0(\alpha), \quad \alpha \in \mathbb{R}^3.$$  \hspace{1cm} (3.3)

The following result shows that the Euler equations, are equivalent to a self-contained evolution equation for the vorticity $\omega$ alone:

**PROPOSITION 3.1** (Vorticity-Stream Formulation in all of $\mathbb{R}^N$). For flows vanishing sufficiently rapidly as $|x| \to \infty$, the Euler equations (3.1) are equivalent to the vorticity-stream formulation,

$$\begin{cases} \frac{D\omega}{Dt} = (\omega \cdot \nabla)u, & (x,t) \in \mathbb{R}^N \times [0,\infty) \\ \omega|_{t=0} = \omega_0, & \end{cases}$$

where the velocity is determined from the vorticity $\omega$ by the Biot-Savart law

$$u(x,t) = \int_{\mathbb{R}^N} K_N(x-y)\omega(y,t) dy, \quad x \in \mathbb{R}^N,$$  \hspace{1cm} (3.4)

involving the kernels $K_N$ homogeneous of degree $1 - N$:

$$K_2(x) = \frac{1}{2\pi} \left( -\frac{x_2}{|x|^2}, \frac{x_1}{|x|^2} \right), \quad x \in \mathbb{R}^2,$$

$$K_3(x)h = \frac{1}{4\pi} \frac{x \times h}{|x|^3}, \quad x, h \in \mathbb{R}^3.$$

Similarly, the pressure $p$ can be obtained from the Poisson equation:

$$-\Delta p = \text{tr} (\nabla u)^2.$$
3.2 Formulation of the Euler equations for the particle trajectories

Our goal now is to derive a second reformulation of the Euler equations as an integro-differential equation for the particle trajectories. Given a smooth velocity field \( u(x, t) \), we discussed in Section 1.3 that the particle trajectories \( X(\alpha, t) \) satisfy

\[
\frac{dX}{dt}(\alpha, t) = u(X(\alpha, t), t), \quad X(\alpha, t)|_{t=0} = \alpha.
\] (3.5)

We showed in Proposition 3.1 that the divergence-free velocity \( u \) can be recovered from the vorticity \( \omega \) by the convolution of the vorticity with a homogeneous kernel \( K_N \). We also know from Chapter 1 that the time-dependent map \( X(\cdot, t) \) connects the Lagrangian reference frame (with variable \( \alpha \)) to the Eulerian reference frame (with variable \( x \)) using the expression: \( x = X(\alpha, t) \). According to Definition 1.4, incompressibility implies \( \det(\nabla_\alpha X(\alpha, t)) = 1 \). Thus, by the change of variables \( x' = X(\alpha', t) \), we rewrite (3.4) in the Lagrangian frame as:

\[
u(X(\alpha, t), t) = \int_{\mathbb{R}^N} K_N(X(\alpha, t) - X(\alpha', t))\omega(X(\alpha', t), t) d\alpha'.
\]

For 3D flows, substituting (3.3) in the previous velocity expression, and using (3.5), we obtain the 3D equation for the particle trajectories:

\[
\frac{dX}{dt}(\alpha, t) = \int_{\mathbb{R}^3} K_3(X(\alpha, t) - X(\alpha', t))\nabla_\alpha X(\alpha', t)\omega_0(\alpha') d\alpha, \quad X(\alpha, t)|_{t=0} = \alpha.
\]

Similarly, in the case of 2D flows, using (3.2), and substituting the new velocity expression in (3.5) we have that the 2D equation for the particle trajectories is

\[
\frac{dX}{dt}(\alpha, t) = \int_{\mathbb{R}^2} K_2(X(\alpha, t) - X(\alpha', t))\omega_0(\alpha') d\alpha, \quad X(\alpha, t)|_{t=0} = \alpha.
\]

**Proposition 3.2.** Let \( u_0(x) \) be a smooth velocity field satisfying \( \nabla \cdot u_0 = 0 \) and \( \omega_0 = \nabla \times u_0 \). Let \( X(\alpha, t) \) be the solution of the equation

\[
\begin{cases}
\frac{dX}{dt}(\alpha, t) = \int_{\mathbb{R}^N} K_N(X(\alpha, t) - X(\alpha', t))\omega(X(\alpha', t), t) d\alpha', \\
X(\alpha, t)|_{t=0} = \alpha.
\end{cases}
\] (3.6)

Define the velocity field \( u \) by

\[
u(x, t) = \int_{\mathbb{R}^N} K_N(x - X(\alpha', t))\omega(X(\alpha', t), t) d\alpha'.
\]

Then, (3.6) for the particle trajectories is equivalent to the Euler equations (3.1), for sufficiently smooth solutions with a rapidly decreasing vorticity.
3.3 Local-in-time existence of solutions

In this section, our goal is to prove the local existence of solutions of the Euler equations (3.1). Taking into account that the 3D case is more complex, we will focus only on the study of these solutions. For 2D flows, the procedure is analogous.

We presented in Proposition 3.2 a reformulation of the Euler equations as an integro-differential equation for the particle trajectories. More precisely,

\[
\begin{aligned}
\frac{dX}{dt}(\alpha, t) &= F(X(\alpha, t)) \\
X(\alpha, t)|_{t=0} &= \alpha,
\end{aligned}
\]  

(3.7)

where \( F \) is a nonlinear and nonlocal operator defined as

\[
F(X(\alpha, t)) = \int_{\mathbb{R}^3} K_3(X(\alpha, t) - X(\alpha', t)) \nabla_\alpha X(\alpha', t) \omega_0(\alpha') d\alpha'.
\]  

(3.8)

The main idea is to view (3.7) as an ODEs problem on an infinite-dimensional Banach space and then, use Picard Theorem on a Banach Space [1] to prove the existence of solutions. Therefore, this will also show the existence of local-in-time solutions of the Euler equations.

**Theorem 3.3 (Picard Theorem on a Banach Space).** Let \( \mathcal{O} \subseteq \mathcal{B} \) be an open subset of a Banach space \( \mathcal{B} \), and let \( F(X) \) be a nonlinear operator satisfying the following criteria:

(i) \( F(X) \) maps \( \mathcal{O} \) to \( \mathcal{B} \).

(ii) \( F(X) \) is locally Lipschitz continuous, i.e., for any \( X \in \mathcal{O} \) there exists \( L > 0 \) and an open neighborhood \( U_X \subset \mathcal{O} \) of \( X \) such that

\[
||F(\tilde{X}) - F(\hat{X})||_{\mathcal{B}} \leq L||\tilde{X} - \hat{X}||_{\mathcal{B}}, \quad \text{for all } \tilde{X}, \hat{X} \in U_X.
\]

Then, for any \( X_0 \in \mathcal{O} \), there exists a time \( T \) such that the ODE

\[
\frac{dX}{dt} = F(X), \quad X|_{t=0} = X_0 \in \mathcal{O}
\]

has a unique (local) solution \( X \in \mathcal{C}^1[(-T,T);\mathcal{O}] \).

Note that the reformulation of the Euler equations in terms of the particle trajectories is necessary in order to use this result, since the term \( u \cdot \nabla \) in the material derivative,

\[
\frac{Du}{Dt} = \frac{\partial u}{\partial t} + (u \cdot \nabla)u,
\]

is an unbounded operator on standard Banach spaces. For instance, if we take the Banach space \( \mathcal{C}^{1,\gamma} \), \( 0 < \gamma < 1 \) with the norm \( |\cdot|_{1,\gamma} \) defined in Section 3.3.2, it is clear that the operator \( F \) defined by \( F(f) = (u \cdot \nabla)f \) cannot be bounded since \( |(u \cdot \nabla)f|_{1,\gamma} = \infty \), for functions \( f \in \mathcal{C}^{1,\gamma} \) such that \( \nabla f \) is not differentiable.

The steps we will follow are: firstly, choose an appropriate Banach space \( \mathcal{B} \), secondly, define an open subset \( \mathcal{O} \subseteq \mathcal{B} \), where things work well and, finally, prove that \( F \) in (3.8) satisfies assumptions (i) and (ii) in Picard Theorem.
3.3.1 Singular Integral Operators

We start by making the following observation: the homogeneous kernels $K_N$ of degree $1-N$, such that

\[
K_2(x) = \frac{1}{2\pi} \left(-\frac{x_2}{|x|^2}, \frac{x_1}{|x|^2}\right), \quad x \in \mathbb{R}^2,
\]

\[
K_3(x)h = \frac{1}{4\pi} \frac{x \times h}{|x|^3}, \quad x, h \in \mathbb{R}^3,
\]

have a singularity in $x = 0$. Thus, the gradient of these functions are only well defined in the distribution sense. Let $P_N(x) = \nabla K_N(x)$.

Any function $P$ that is homogeneous of degree $-N$ and has mean-value zero on the unit sphere, i.e.,

\[
\int_{|x|=1} P \, ds = 0, \quad \text{(3.11)}
\]

defines a singular integral operator (SIO) through the convolution

\[
Pf(x) = \text{PV} \int_{\mathbb{R}^N} P(x - y)f(y) \, dy = \lim_{\epsilon \to 0} \int_{|x-y| > \epsilon} P(x - y)f(y) \, dy, \quad \text{(3.12)}
\]

where PV stands for principal value. We claim that (3.11) is a sufficient condition for the limit in (3.12) to exist [3].

For example, if $N = 2$, we have that:

\[
P_2(x) = \nabla K_2(x) = \frac{1}{2\pi} \nabla \left(-\frac{x_2}{|x|^4}, \frac{x_1}{|x|^4}\right) = \left(\frac{2x_1 x_2}{|x|^4} - \frac{x_2^2}{|x|^4}, \frac{x_2^2 - x_1^2}{|x|^4} - \frac{x_1 x_2}{|x|^4}\right).
\]

It is easy to see that $P_2(\lambda x) = \lambda^{-2}P_2(x)$, for $\lambda > 0$ and $x \neq 0$. Hence, $P_2$ is a homogeneous function of degree -2. Moreover, if $P_2 = (p_{ij}(x))_{i,j=1,2}$, then

\[
\int_{|x|=1} p_{11} \, ds = \int_0^{2\pi} p_{11}(\cos \theta, \sin \theta) \, d\theta = \int_0^{2\pi} 2\sin \theta \cos \theta \, d\theta = \int_0^{2\pi} \sin 2\theta \, d\theta = 0
\]

\[
\int_{|x|=1} p_{21} \, ds = \int_0^{2\pi} p_{21}(\cos \theta, \sin \theta) \, d\theta = \int_0^{2\pi} (\cos^2 \theta - \sin^2 \theta) \, d\theta = \int_0^{2\pi} \cos 2\theta \, d\theta = 0.
\]

Since $p_{22}(x) = -p_{11}(x)$ and $p_{12}(x) = -p_{21}(x)$ we deduce that $P_2$ has mean-value zero on the unit sphere. The result also holds for $N = 3$. Therefore, $P_N$ defines the principal-value SIO

\[
P_N f(x) = \text{PV} \int_{\mathbb{R}^N} P_N(x - y)f(y) \, dy.
\]
3.3. Local-in-time existence of solutions

Good regularity conditions on $f$ together with cancellation properties on the singular kernel $P_N$ imply potential theory estimates. Before we see this, we first make a brief introduction to the Hölder spaces. The Hölder–$\gamma$ norm is defined by

$$
\|X\|_\gamma = |X|_0 + |X|_\gamma, \quad 0 < \gamma < 1,
$$

where $|\cdot|_0$ is the supremum norm given by

$$
|X|_0 = \sup_{\alpha \in \mathbb{R}^N} |X(\alpha)|,
$$

and $|\cdot|_\gamma$ is the Hölder seminorm

$$
|X|_\gamma = \sup_{\alpha, \alpha' \in \mathbb{R}^3, \alpha \neq \alpha'} \frac{|X(\alpha) - X(\alpha')|}{|\alpha - \alpha'|^\gamma}, \quad 0 < \gamma < 1.
$$

Any function $X : \mathbb{R}^N \rightarrow \mathbb{R}^N$ satisfying $|X|_\gamma < \infty$ is called Hölder continuous. We denote by $C^\gamma$ the space of functions with bounded $|\cdot|_\gamma$ norm. A very important result for our interest is that SIOs are bounded operators on the Hölder spaces.

**Lemma 3.4.** Let $K_N$ be the integral operator defined by

$$
K_N f(x) = \int_{\mathbb{R}^N} K_N(x - y) f(y) \, dy,
$$

where the kernel $K_N$ is smooth outside $x = 0$ and homogeneous of degree $1 - N$. Consider $f \in C^\gamma(\mathbb{R}^N; \mathbb{R}^N)$, $0 < \gamma < 1$ with compact support and define $m_f = m(\text{supp} f) < \infty$. Denote $m_f = R_N$ for some $R > 0$. Then, there exists a constant $c$, independent of $f$ and $R$, so that

$$
|K_N f|_0 \leq c R |f|_0.
$$

**Lemma 3.5.** Let $f$, $\gamma$, and $R$ satisfy the assumptions of Lemma 3.4. Let the SIO $P_N = \nabla K_N$ satisfy:

(i) $P_N$ is homogeneous of degree $-N$,

(ii) $P_N$ has mean-value zero on the unit sphere,

(iii) $P_N f(x) = \text{PV} \int_{\mathbb{R}^N} P_N(x - y) f(y) \, dy$.

Then, there exists a constant $c$, independent of $R$ and $f$, so that

$$
|P_N f|_0 \leq \left\{ |f|_\gamma \epsilon^\gamma + \max \left( 1, \ln \frac{R}{\epsilon} \right) |f|_0 \right\}, \quad \forall \epsilon > 0,
$$

$$
|P_N f|_\gamma \leq c \|f\|_\gamma.
$$
Chapter 3. The Particle-Trajectory Method

We have seen that our kernels $K_N$ defined in equations (3.9) and (3.10) satisfy the assumptions of Lemmas 3.4 and 3.5. Therefore, we have useful estimates for the integral operators $K_N f$ and $P_N f$. For instance, consider our homogeneous kernel for $N = 2$,

$$K_2(x) = \frac{1}{2\pi} \left(-\frac{x_2}{|x|^2}, \frac{x_1}{|x|^2} \right) = \frac{\Omega(x')}{|x|},$$

where $x' = x/|x|$ and $\Omega : S^1 \to \mathbb{R}^2$ is a continuous function on $S^1$ defined by

$$\Omega(x') = \frac{1}{2\pi} (-x'_2, x'_1).$$

Note that

$$|\Omega|_0 = \sup_{x' \in S^1} |\Omega(x')| = \frac{1}{2\pi}. \quad (3.13)$$

Hence, under the assumptions of Lemma 3.4, it holds that

$$\left| \int_{\mathbb{R}^2} K_2(x - y) f(y) \, dy \right| \leq \int_{\mathbb{R}^2} |K_2(z)||f(x - z)| \, dz$$

$$= \int_{|z| < R} \frac{|\Omega(z')|}{|z|}|f(x - z)| \, dz + \int_{|z| \geq R} \frac{|\Omega(z')|}{|z|}|f(x - z)| \, dz$$

$$= I + II.$$

Using (3.13) we have that

$$I \leq \frac{1}{2\pi} |f|_0 \int_0^R 2\pi \frac{1}{\rho} \, d\rho = R|f|_0.$$

$$II \leq \frac{1}{2\pi} |f|_0 \int_{|z| \geq R} \frac{1}{|z|} \, dz \leq \frac{1}{2\pi} |f|_0 \frac{1}{R} \mathcal{M}_f = \frac{1}{2\pi} R|f|_0.$$

Then,

$$|K_2 f|_0 \leq \left(1 + \frac{1}{2\pi} \right) R|f|_0.$$

### 3.3.2 The choice of the Banach space and the open subset

We return now to the problem of choosing a complete normed infinite Banach space $B$ and an open subset $O \subseteq B$ such that $F : O \to B$, given by the expression,

$$F(X(\alpha, t)) = \int_{\mathbb{R}^3} K_3(X(\alpha, t) - X(\alpha', t))\nabla_\alpha X(\alpha', t)\omega_0(\alpha') \, d\alpha',$$

is bounded with respect to the norm in the space and locally Lipschitz continuous. Recall from Section 1.2 that the particle-trajectory maps

$$X_t = X(\cdot, t) : \mathbb{R}^3 \to \mathbb{R}^3, \quad t \in [0, T]: T > 0$$
are diffeomorphisms. Thus, our open set \( \mathcal{O} \) must contain differentiable, 1-1 and onto functions, with differentiable inverse function. Observe that the identity map \( X_0 \) must also be in \( \mathcal{O} \). Furthermore, since the gradient of \( F \) depends on the term \( \nabla_\alpha X \), as well as a SIO, \( P_3(X) = \nabla K_3(x) \), which is bounded on \( C^7 \), then \( \mathcal{B} \) must contain all functions \( X : \mathbb{R}^3 \rightarrow \mathbb{R}^3 \) for which \( \nabla_\alpha X \) is Hölder continuous. Thus, we define the Banach space \( \mathcal{B} \) as

\[
\mathcal{B} = \{ X : \mathbb{R}^3 \rightarrow \mathbb{R}^3 \text{ such that } |X|_{1,\gamma} < \infty \},
\]

where \( | \cdot |_{1,\gamma} \) is the norm given by

\[
|X|_{1,\gamma} = |X(0)| + |\nabla X(0)|_0 + |\nabla_\alpha X|_\gamma = |X(0)| + \|\nabla_\alpha X\|_\gamma.
\]

The condition \( |X(0)| < \infty \) is necessary for \( | \cdot |_{1,\gamma} \) to define a norm and it also allows \( X \) to be unbounded as \( |\alpha| \nearrow \infty \). On the other hand, we will show that since \( |\nabla_\alpha X|_\gamma < \infty \), then \( F \) is bounded and Lipschitz continuous on a subset of \( \mathcal{B} \). In order to choose this appropriate open subset we need to add an additional condition forced by the incompressibility of the flow, which is \( \det \nabla_\alpha X(\alpha) = 1 \). However, this condition is too restrictive because it defines a hypersurface of functions in \( \mathcal{B} \). Instead, we define

\[
\mathcal{O}_M = \left\{ X \in \mathcal{B} : \inf_{\alpha \in \mathbb{R}^3} \det \nabla_\alpha X(\alpha) > \frac{1}{2} \text{ and } |X|_{1,\gamma} < M \right\}.
\]

Note that the identity map belongs to \( \mathcal{O}_M \), for \( M > 1 \). Furthermore, if \( X \in \mathcal{O}_M \), then \( \det \nabla_\alpha X(\alpha) \neq 0 \) and by the Inverse Function Theorem, we have that \( X \) is a local homeomorphism. However, because particle-trajectory maps are global homeomorphisms, we need a stronger condition over the mappings in \( \mathcal{O}_M \). Hadamard Lemma will give us the result.

**Lemma 3.6 (Hadamard).** Suppose that \( X \in \mathcal{B} \) is a local homeomorphism, and there exists \( c > 0 \) such that

\[
\sup_{\alpha \in \mathbb{R}^3} |(\nabla_\alpha X)^{-1}(\alpha)| \leq c.
\]

Then, \( X \) is a homeomorphism of \( \mathbb{R}^3 \) onto \( \mathbb{R}^3 \).

Before we show that the family of open subsets \( \{ \mathcal{O}_M \}_{M>1} \) is well defined for our purpose, we state the following norm inequalities [6]:

**Lemma 3.7.** Let \( X : \mathbb{R}^N \rightarrow \mathbb{R}^N \) be a smooth invertible transformation with

\[
|\det \nabla_\alpha X(\alpha)| \geq c_1 > 0.
\]

Then, for \( 0 < \gamma < 1 \) there exists \( c > 0 \) such that

\[
\| (\nabla_\alpha X)^{-1} \|_{\gamma} \leq c \| \nabla_\alpha X \|_{2^N-1}\gamma
\]

\[
|X^{-1}|_{1,\gamma} \leq c |X|_{1,\gamma}^{2^N-1}.
\]
Lemma 3.8. Let $X: \mathbb{R}^N \to \mathbb{R}^N$ be an invertible transformation with
\[ |\det \nabla \alpha X(\alpha)| \geq c_1 > 0, \]
and let $f: \mathbb{R}^N \to \mathbb{R}^M$ be a smooth function. Then, for $0 < \gamma < 1$,
\[ \|f \circ X^{-1}\|_\gamma \leq \|f\|_\gamma (1 + c|X|_{1,\gamma}^{(2N-1)}). \]

Proposition 3.9. For any $M > 1$ and $0 < \gamma < 1$, the subset $O_M \subseteq B$ is nonempty, open, and it consists of homeomorphisms of $\mathbb{R}^3$ onto $\mathbb{R}^3$.

Proof. Given any $M > 1$, the subset
\[ O_M = \left\{ X \in B : \inf_{\alpha \in \mathbb{R}^3} \det \nabla \alpha X(\alpha) > \frac{1}{2} \text{ and } |X|_{1,\gamma} < M \right\} \subseteq B \]
is nonempty because for all $M > c \geq 1$,
\[ c|I|_{1,\gamma} = |cI(0)| + c|\nabla \alpha I|_0 + c|\nabla \alpha I|_{\gamma} = c < M, \]
where $I$ is the identity map in $\mathbb{R}^3$, and $\det \nabla \alpha cI(\alpha) = c^3 > 1/2$, for all $\alpha \in \mathbb{R}^3$. Thus, $cI \in O_M$. Moreover, the map $\inf_{\alpha \in \mathbb{R}^3} \det \nabla \alpha : B \to \mathbb{R}^3$ and the norm $|\cdot|_{1,\gamma} : B \to \mathbb{R}^3$ are continuous. Therefore, inverse images of the open subsets $(1/2, +\infty)$ and $[0, M)$ are open subsets in the Banach space $B$. Thus, $O_M$ is open in $B$. Finally, by the definition of $O_M$, we have that $\det \nabla \alpha X(\alpha) > 1/2$ and $|X|_{1,\gamma} < M$, and using Lemma 3.7
\[ |(\nabla \alpha X)^{-1}(\alpha)| \leq |(\nabla \alpha X)^{-1}|_0 \leq \|(\nabla \alpha X)^{-1}\|_\gamma \leq c\|\nabla \alpha X\|_\gamma^5, \quad \forall \alpha \in \mathbb{R}^3, \]
we obtain
\[ \sup_{\alpha \in \mathbb{R}^3} |(\nabla \alpha X)^{-1}(\alpha)| \leq c\|\nabla \alpha X\|_\gamma^5 \leq c|X|_{1,\gamma}^5 \leq cM^5. \]

Then, applying Hadamard Lemma 3.6, $X: \mathbb{R}^3 \to \mathbb{R}^3$ is an homeomorphism, for all $X \in O_M$, and we have the result.

3.3.3 Local existence and uniqueness of solutions

The main goal of this section is to prove the following result that provides the local-in-time existence and uniqueness of solutions of the Euler equations. The assumption on the size of support of $\omega_0$ is not necessary to prove the theorem, but it is useful to simplify the technical arguments in the potential theory estimates. The proof follows directly from Picard Theorem 3.3 and thus, we only need to show that the required assumptions are satisfied.

Theorem 3.10. Consider a compactly supported initial vorticity $\omega_0 \in \mathcal{C}^\gamma$, $\gamma \in (0, 1)$, $\omega_0 = \nabla \times v_0$ and $\nabla \cdot v_0 = 0$. Then, for any $M > 1$ there exists $T(M) > 0$ and a unique volume-preserving solution
\[ X \in \mathcal{C}^1((-T(M), T(M)); O_M) \]
to the particle-trajectory equations (3.7) and (3.8).
We have seen that the integro-differential equations for the particle trajectories

\[
\begin{align*}
\frac{dX}{dt}(\alpha, t) &= F(X(\alpha, t)) \\
X(\alpha, t)|_{t=0} &= \alpha,
\end{align*}
\]

form an ODEs problem on the Banach space

\[ B = \{ X : \mathbb{R}^3 \rightarrow \mathbb{R}^3 \text{ such that } |X|_{1, \gamma} < \infty \}, \]

where \( | \cdot |_{1, \gamma} \) is the norm defined by

\[ |X|_{1, \gamma} = |X(0)| + |\nabla_\alpha X|_0 + |\nabla X|_\gamma. \]

Our next step is to show that \( F \) is a bounded, nonlinear, Lipschitz continuous operator on the Banach space \( B \).

**Proposition 3.11.** Consider \( \omega_0 \in C^\gamma(\mathbb{R}^3; \mathbb{R}^3), \ 0 < \gamma < 1. \) Let \( F : \mathcal{O}_M \rightarrow B \) be defined by

\[ F(X(\alpha, t)) = \int_{\mathbb{R}^3} K_3(X(\alpha, t) - X(\alpha', t))\nabla_\alpha X(\alpha', t)\omega_0(\alpha') \, d\alpha'. \]

Then, \( F \) satisfies assumptions of the Picard theorem, i.e., \( F \) is bounded and locally Lipschitz continuous on \( \mathcal{O}_M \).

**Proof.** First, we prove that \( F \) is a bounded operator, that is

\[ |F(X)|_{1, \gamma} < \infty, \quad \forall X \in \mathcal{O}_M. \]

Recall from Proposition 3.9 that the open subset \( \mathcal{O}_M \) consists of homeomorphisms \( X : \mathbb{R}^3 \rightarrow \mathbb{R}^3 \). Then, we use the change of variables \( X^{-1}(x) = \alpha \) and rewrite \( F(X) \) as

\[ F(X) = K_3f \circ X^{-1}, \]

\[ K_3f(x) = \int_{\mathbb{R}^3} K_3(x - x')f(x') \, dx', \]

\[ f(x') = \nabla_\alpha X(\alpha\omega_0(\alpha)|_{\alpha=X^{-1}(x')} \det \nabla_x X^{-1}(x'). \quad (3.14) \]

Calculus inequality in Lemma 3.7 implies that

\[ |F(X)|_{1, \gamma} \leq |K_3f|_{1, \gamma}|X^{-1}|_{1, \gamma} \leq c|K_3f|_{1, \gamma}|X|_{1, \gamma}^5 < cM^5|K_3f|_{1, \gamma}, \]

so we only need to estimate \( |K_3f|_{1, \gamma} \).

Because we want to use potential theory estimates in Lemmas 3.4 and 3.5, the function \( f \) in (3.14) has to be in the Hölder \( \gamma \)-space, \( C^\gamma \). Namely, using Lemma 3.8 for the composition \( (\nabla_\alpha X\omega_0) \circ X^{-1} \), we have that
\[ \|f\|_\gamma \leq \|\nabla_a X \omega_0\| X^{-1}\|_\gamma \|\nabla_x X^{-1}\|_\gamma \]
\[ \leq \|\nabla_a X \omega_0\|_\gamma (1 + c|X|^{2N-1}) \|\nabla_x X^{-1}\|_\gamma \]
\[ \leq \|\nabla_a X\|_\gamma \|\omega_0\|_\gamma (1 + c|X|^{2N-1}) \|(\nabla_a X)^{-1}\|_\gamma. \]

We then use Lemma 3.7 for the estimation of the last term:
\[ \|(\nabla_a X)^{-1}\|_\gamma \leq c\|\nabla_a X\|^2 \leq c|X|^{2N-1}. \]

Hence, since \( X \in O_M \) implies that \( |X|_{1,\gamma} < M \), we get that
\[ \|f\|_\gamma \leq c(M)\|\omega_0\|_\gamma, \]
and, consequently, \( f \in C^\gamma \).

We want to estimate \( |K_3 f|_{1,\gamma} = |K_3 f(0)| + \|\nabla K_3 f\|_\gamma \). By Lemma 3.4,
\[ |K_3 f(0)| \leq |K_3 f|_0 \leq cR|f|_0, \tag{3.15} \]
where \( R^N = m(\text{supp} f) \). For the second term, it can be proved that [6]
\[ \nabla[K_N f(x)] = \text{PV} \int_{R^N} P_N(x - x') f(x') \, dx' + cf(x), \]
where \( P_N(x) = \nabla K_N(x) \) satisfies the assumptions in Lemma 3.5. Then,
\[ \|\nabla K_3 f\|_\gamma \leq c\|f\|_\gamma + \|P_3 f\|_\gamma \leq c\|f\|_\gamma \leq c(M)\|\omega_0\|_\gamma. \tag{3.16} \]

Combining norm inequalities (3.15) and (3.16), we get that \( F: O_M \to B \) is bounded. It remains to show that \( F \) is locally Lipschitz continuous on \( O_M \). It suffices to prove that the derivative \( F'(X) \) is bounded as a linear operator from \( O_M \) to \( B \). We compute the derivative as
\[
F'(X)Y = \frac{d}{d\epsilon} F(X + \epsilon Y)|_{\epsilon=0} \\
= \frac{d}{d\epsilon} \int_{\mathbb{R}^3} K_3(X(\alpha) - X(\alpha') + \epsilon(Y(\alpha) - Y(\alpha'))) \\
\cdot \nabla_a (X(\alpha') + \epsilon Y(\alpha'))\omega_0(\alpha') \, d\alpha'|_{\epsilon=0} \\
= \int_{\mathbb{R}^3} K_3(X(\alpha) - X(\alpha'))\nabla_a Y(\alpha')\omega_0(\alpha') \, d\alpha' \\
+ \int_{\mathbb{R}^3} \nabla K_3(X(\alpha) - X(\alpha'))(Y(\alpha) - Y(\alpha'))\nabla_a X(\alpha')\omega_0(\alpha') \, d\alpha' \\
\equiv G_1(X)Y + G_2(X)Y.
\]
3.4 Global-in-time existence of smooth solutions

Note that, proceeding in the same manner as before, we can rewrite \(G_1(X)Y\) as the following convolution operator

\[
G_1(X)Y = K_3 g_1(x),
\]

\[
g_1(x) = \nabla_\alpha Y(\alpha) \omega_0(\alpha)|_{\alpha = X^{-1}(x')} \det \nabla_x X^{-1}(x'),
\]

and again Lemma 3.4 provides the following estimate

\[
|G_1(X)Y|_{1,\gamma} \leq c\|g_1\|_{\gamma} \leq c\|\nabla_\alpha Y \omega_0 \circ X^{-1}\|_{\gamma} \|\nabla_x X^{-1}\|_{\gamma}.
\]

Calculus inequalities in Lemmas 3.7 and 3.8 imply that

\[
|G_1(X)Y|_{1,\gamma} \leq c\|\nabla_\alpha Y\|_{\gamma} \|\omega_0\|_{\gamma} (1 + c\|X^{(2N-1)}\|_{1,\gamma}^{(2N-1)} \leq c(M)\|\omega_0\|_{\gamma} Y|_{1,\gamma}.
\]

Observe now that \(G_2(X)Y\) is not a SIO of the form \(P_3 g_2(x)\), for any \(g_2(x)\), because the term \(Y(\alpha) - Y(\alpha')\) kills the singularity of \(\nabla K_3\). Therefore, we cannot use potential estimates of Lemma 3.5. The estimation of \(G_2(X)\) is quite technical so we will only state the final result (the details may be found in [6]):

\[
|G_2(X)Y|_{1,\gamma} \leq c(M)\|\omega_0\|_{\gamma} Y|_{1,\gamma}.
\]

By the Fundamental Theorem of Calculus and the Minkowski inequality [8],

\[
|F(\tilde{X}) - F(\hat{X})|_{1,\gamma} = \left| \int_0^1 \frac{d}{d\epsilon} F(\tilde{X} + \epsilon(\hat{X} - \tilde{X})) d\epsilon \right|_{1,\gamma}
\]

\[
\leq \left( \int_0^1 \|F'(\tilde{X} + \epsilon(\hat{X} - \tilde{X}))\| d\epsilon \right) |\tilde{X} - \hat{X}|_{1,\gamma}
\]

\[
\leq L |\tilde{X} - \hat{X}|_{1,\gamma},
\]

where \(\|\cdot\|\) denotes the norm of the derivative operator and \(L\) is a positive constant because \(F'(X) : \mathcal{O}_M \rightarrow \mathcal{B}\) is a bounded operator. Hence, \(F\) is locally Lipschitz continuous on the Banach space \(\mathcal{B}\).

Our last goal in the next section will be to discuss the possible existence and uniqueness of global solutions of the Euler equations. We will see that vorticity plays an important role and therefore, results will differ from the 2D and the 3D case.

3.4 Global-in-time existence of smooth solutions

We have seen in the previous section that Picard Theorem 3.3 provides a locally in time unique solution \(X(\alpha, t)\) to the integro-differential equations for the particle trajectories. Recall from Section 2.5 that vorticity is transported and stretched along particle trajectories for 3D flows and it is conserved along particle paths for 2D flows. The key point is that if the magnitude of the vorticity remains bounded, then the solution will exist globally in time.

This result is due to Beale, Kato and Majda (1984) and it provides a sufficient condition, which determines when it is possible to continue the solution \(X(\alpha, t)\) further in time.
Theorem 3.12 (Beale-Kato-Majda). Consider a compactly supported initial vorticity \( \omega_0 = \nabla \times v_0, \nabla \cdot v_0 = 0 \), with Hölder norm \( \| \omega_0 \|_\gamma < \infty \) for some \( \gamma \in (0, 1) \). Let \( |\omega(\cdot, s)|_0 \) be the supremum norm at fixed time \( s \) of a solution to the Euler equation, with initial data \( \omega_0 \).

(i) Suppose that for any \( T > 0 \) there exists \( M_1 > 0 \) such that vorticity \( \omega(x, t) \) satisfies

\[
\int_0^T |\omega(\cdot, s)|_0 \, ds \leq M_1.
\]

Then, for any \( T \) there exists \( M > 0 \) such that \( X \in C^1([0, T); O_M) \), i.e., the solution exists globally in time.

(ii) Suppose that for any \( M > 0 \) there is a finite maximal time \( T(M) \) of existence of solutions \( X \in C^1([0, T(M)); O_M) \) and that \( \lim_{M \to \infty} T(M) = T^* < \infty \); then necessarily the vorticity accumulates so rapidly that

\[
\lim_{t \to T^*} \int_0^t |\omega(\cdot, s)|_0 \, ds = \infty.
\]

As a consequence of Theorem 3.12 we have that the 2D Euler equations have global-in-time existence of solutions because vorticity does not grow in time and hence, it cannot become unbounded in magnitude.

Corollary 3.13 (Global Existence of Solutions to the Euler Equations in Two Dimensions). Consider the 2D Euler equations with compactly supported initial vorticity \( \omega_0 = \nabla \times v_0, \nabla \cdot v_0 = 0 \) satisfying \( \| \omega_0 \|_\gamma < \infty \) for some \( \gamma \in (0, 1) \). Then there exists a unique solution \( X(\cdot, t) \in C^1((-\infty, \infty); B) \) to the particle-trajectory equation in the time interval \((-\infty, \infty)\).

The proof of Theorem 3.12 uses as a key point the fact that the operator \( F \) is a Lipschitz function independent of time. For this particular case, Ladas and Lakshmikantham (1972) state in the following theorem that the ability to continue the local solution in time is directly related with the absence of blowup in the norm \(|\cdot|_{1,\gamma}\).

Theorem 3.14 (Continuation of an Autonomous ODE on a Banach Space). Let \( O \subset B \) be an open subset of a Banach space \( B \), and let \( F : O \to B \) be a locally Lipschitz continuous operator. Then the unique solution \( X \in C^1([0, T); O) \) to the autonomous ODE,

\[
\frac{dX}{dt} = F(X), \quad X|_{t=0} = X_0 \in O,
\]

either exists globally in time or \( T < \infty \) and \( X(t) \) leaves the open set \( O \) as \( t \nearrow T \).

Because the particle trajectories are volume preserving, the only way the solution can leave the set \( O_M \) is if \(|X|_{1,\gamma}\) grows larger that \( M \). Hence, we obtain a sufficient condition for global-in-time existence by finding a sufficient condition for \(|X|_{1,\gamma}\) to be a priori bounded.
3.5 Classical solutions of the Euler equations

We end up this chapter by giving some examples of the exact solutions in Equations (3.1) for 3D flows, to illustrate all the above theory presented [2].

Example 3.15 (Stationary solutions). We take $\omega_0 = \nabla \times u_0 = 0$. Then,
\[
    u(x,t) = (\gamma_1 x_1, \gamma_2 x_2, -(\gamma_1 + \gamma_2)x_3),
\]
\[
    p(x,t) = -\frac{\gamma_1}{2}x_1^2 - \frac{\gamma_2}{2}x_2^2 - \frac{\gamma_3}{2}x_3^2,
\]
is a solution of the Euler equations, for certain $\gamma_j > 0, j = 1, 2, 3$. Observe that the velocity field $u$ and the scalar pressure $p$ are independent of time. However, the particle trajectories are
\[
    X(\alpha,t) = (\alpha_1 e^{\gamma_1 t}, \alpha_2 e^{\gamma_2 t}, \alpha_3 e^{\gamma_3 t}),
\]
where $\alpha = (\alpha_1, \alpha_2, \alpha_3) \in \mathbb{R}^3$. Furthermore, this flow is irrotational: $\omega = \nabla \times u = 0$.

Example 3.16 (Solutions with a singularity). Let $T^*$ denote the blowup time in Theorem 3.12. Then, the solutions of the Euler equations
\[
    u = \left( -\frac{x_1}{T^*-t}, \frac{x_2}{T^*-t}, 0 \right),
\]
\[
    p = \frac{x_2^2}{(T^*-t)^2},
\]
develop a singularity for $t \to T^*$. In fact, it is still not known whether solutions to 3D Euler equations can develop singularities in finite time.

Example 3.17 (Axisymmetric solutions). In cylindrical coordinates,
\[
    u(x,t) = u^\rho(r,x_3,t) e_r + u^\theta(r,x_3,t) e_\theta + u^3(r,x_3,t) e_3
\]
is a 3D axisymmetric exact solution of the Euler equations, where $e_r = (\frac{x_1}{r}, \frac{x_2}{r}, 0), e_\theta = (-\frac{x_2}{r}, \frac{x_1}{r}, 0), e_3 = (0,0,1)$ and $r = (x_1^2 + x_2^2)^{\frac{1}{2}}$. Note that the velocity field only depends on two space coordinates, $r$ and $x_3$. The velocity $u^\theta$ in the $e_\theta$ direction is called the swirl velocity.

Axisymmetric flows are of particular interest since they define an intermediate case between the 2D and the 3D flows. The case of axisymmetric flows without swirl, i.e., $u^\theta = 0$, is similar to that of 2D flows, because the solution exists globally in time. However, axisymmetric flows with swirl have a nontrivial 3D behavior. In fact, the existence of global solutions when $u^\theta \neq 0$ is currently an open problem.
Bibliography


