2021 Texas Putnam Prep Group — week 1

Welcome to our Putnam Exam preparation meetings. We'd like to help you do well with Putnam Competition problems. There's no better way to do so than to try solving hard problems, and then presenting solutions to each other and critiquing others' solutions. So without further ado, let's see some problems!

1. For any positive integer k let $f(k)$ denote the sum of the squares of the digits of k when k is written in ordinary decimal notation. Then define functions f_n by $f_1 = f$ and for $n > 1$, $f_n = f \circ f_{n-1}$. So for example, $f_1(123) = f(123) = 1^2 + 2^2 + 3^2 = 14$; $f_2(123) = f(f_1(123)) = f(14) = 17$; $f_3(123) = f(17) = 50$.

Compute $f_{2021}(11)$.

2. For any integer n let $c(n)$ count the number of ways to express n as a sum of k positive integers a_i which satisfy $a_1 \le a_2 \le a_3 \dots a_{k-1} \le a_k \le a_1 + 1$. For example for $n = 4$ there are four such representations:

$$4 = 4 \qquad 4 = 2 + 2 \qquad 4 = 1 + 1 + 2 \qquad 4 = 1 + 1 + 1 + 1$$

so $c(4) = 4$. Find (with proof) a formula to compute $c(n)$ for all n.

3. Show that for every set of n real numbers $x_i \in [0, 1]$ we have

$$(1 + x_1)(1 + x_2) \dots (1 + x_n) \le 2^{n-1}(1 + x_1 x_2 \dots x_n)$$

4. Evaluate the determinant

$$\det \begin{vmatrix} \cos(1) & \cos(2) & \cos(3) \\ \cos(4) & \cos(5) & \cos(6) \\ \cos(7) & \cos(8) & \cos(9) \end{vmatrix}$$

5. Prove that if a, b, c are integers and $a + b\sqrt{2} + c\sqrt{3} = 0$ then $a = b = c = 0$.

6. Show that for every cubic polynomial $p(x)$ there is a straight line that meets the graph of $y = p(x)$ in three distinct points.

And the question that may have drawn you here tonight (from last year's Putnam exam):

0. Define a sequence a_n by $a_0 = \frac{\pi}{2}$ and for $n > 0$ we have $a_n = \sin(a_{n-1})$. Does the series $\sum_{n \ge 0} a_n^2$ converge?