UT Putnam Prep Problems, Nov 11 2021
SOME LINEAR-ALGEBRA PUTNAM PROBLEMS

1. Suppose $A, B \in M_4(\mathbf{R})$ commute, and $\det(A^2 + AB + B^2) = 0$. Prove that

$$\det(A + B) + 3\det(A - B) = 6\det(A) + 6\det(B).$$

2. (10B1) Is there an infinite sequence of real numbers $a_1, a_2, a_3, \ldots$ such that

$$a_1^m + a_2^m + a_3^m + \cdots = m$$

for every positive integer m?

3. (95A5) Let $x_1, x_2, \ldots, x_n$ be differentiable (real-valued) functions of a single variable f which satisfy

$$\frac{dx_1}{dt} = a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n$$
$$\frac{dx_2}{dt} = a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n$$
$$\cdots$$
$$\frac{dx_n}{dt} = a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n$$

for some constants $a_{ij} > 0$. Suppose that for all i, $x_i(t) \to 0$ as $t \to \infty$. Are the functions $x_1, x_2, \ldots, x_n$ necessarily linearly dependent?

4. (95A6) Suppose that each of n people writes down the numbers 1,2,3 in random order in one column of a $3 \times n$ matrix, with all orders equally likely and with the orders for different columns independent of each other. Let the row sums a, b, c of the resulting matrix be rearranged (if necessary) so that $a \leq b \leq c$. Show that for some $n \geq 1995$, it is at least four times as likely that both $b = a + 1$ and $c = a + 2$ as that $a = b = c$.

5. Suppose $A \in M_n(\mathbf{C})$ has rank r, where $1 \leq r \leq n - 1$ and $n > 1$. Show that there exist matrices $B \in M_{n,r}(\mathbf{C})$ and $C \in M_{r,n}(\mathbf{C})$ with $A = BC$.

6. (Problem 2008-A-2). Alan and Barbara play a game in which they take turns filling entries of an initially empty 2008×2008 array. Alan plays first. At each turn, a player chooses a real number and places it in a vacant entry. The game ends when all the entries are filled. Alan wins if the determinant of the resulting matrix is nonzero; Barbara wins if its is zero. Which player has a winning strategy?

7. (1990-A-5). If A and B are square matrices of the same size such that $ABAB = 0$, does it follow that $BABA = 0$?

8. (1994-A-4). Let A and B be 2×2 matrices with integer entries such that A, $A + B$, $A + 2B$, $A + 3B$, and $A + 4B$ are all invertible matrices whose inverses have integer entries. Show that $A + 5B$ is invertible and that its inverse has integer entries.

Now flip over for some additional practice with Axiomatic Mathematics!

BONUS ROUND! A vector space may be defined as a set V on which two binary operations called $+$ and $\cdot$ are defined (respectively as functions $V \times V \to V$ and $\mathbf{R} \times V \to V$) subject to a set of axioms. We may express these axioms in the following way:

$\mathtt{VS}_1$. For all $u, v, w \in V$ we have $u + (v + w) = (u + v) + w$

$\mathtt{VS}_2$. For all $u, v \in V$ we have $u + v = v + u$

$\mathtt{VS}_3$. There is a vector $u \in V$ so that for all $v \in V$ we have $u + v = v$

$\mathtt{VS}_4$. For all $u, v, w \in V$, if $u + w = v + w$ then $u = v$; likewise if $w + u = w + v$ then $u = v$.

$\mathtt{VS}_5$. For all $u, v \in V$ and all $a \in \mathbf{R}$ we have $a \cdot (u + v) = a \cdot u + a \cdot v$

$\mathtt{VS}_6$. For all $u \in V$ and all $a, b \in \mathbf{R}$ we have $(a + b) \cdot u = a \cdot u + b \cdot u$

$\mathtt{VS}_7$. For all $u \in V$ and all $a, b \in \mathbf{R}$ we have $(ab) \cdot u = a \cdot (b \cdot u)$

$\mathtt{VS}_8$. For all $u \in V$ we have $1 \cdot u = u$

For each of these axioms, give an example of an object which satisfies all the axioms EXCEPT the given one, that is, a non-vector space that satisfies the other seven axioms.

Here's an example. Take the set V to be the set of real numbers; define "vector addition" on V to be ordinary addition of real numbers; and define "scalar multiplication" by

$$c \cdot v = 0 \qquad \text{for all scalars } c \text{ and vectors } v$$

Then axioms $\mathtt{VS}_1$ through $\mathtt{VS}_7$ are satisfied but axiom $\mathtt{VS}_8$ is not. Your job is to construct other examples (saying exactly what V, "$+$", and "$\cdot$" are) where seven of the axioms are satisfied but the remaining one is not. (I'm looking for one example where $\mathtt{VS}_1$ is violated, another where $\mathtt{VS}_2$ is violated, etc.)

This can be done for seven of the axioms, but one of these axioms is actually redundant — it automatically follows from the other seven axioms. Which of the eight axioms is redundant?