HAMILTON'S PRINCIPLE: minimality of the action integral.

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INTRODUCTION: When an object moves under the influence of some force, the motion is described by Newton's famous law F = ma. If the force happens to come from a potential field (such as is true with gravity), then at any instant of time t, the object incorporates both some kinetic energy K(t) due to its motion, and some potential energy P(t) due to its position. It is well known (and proven herein) that K(t) + P(t) is a constant, that being a special case of the law of conservation of energy. Less well known is that K(t) – P(t) also has an interesting property. Suppose the particle starts at point A at time t = 0 and ends at point B at time t = T. Suppose one considers all parameterized curves $\sigma(t)$ with $\sigma(0) = A$ and $\sigma(T) = B$. Suppose over each such curve, one calculates the integral of K(t) – P(t) (for $0 \le t \le T$). It will be shown that for T sufficiently small, that integral is a minimum when the curve $\sigma(t)$ is the actual trajectory of the object. In other words, when it is the trajectory determined by Newton's law.

In section 1, we consider the very special case that there is no potential field (and hence no force) present, and we show that linear motion at constant speed minimizes the integral of K(t) (noting that P(t) = 0 in the absence of a potential field.) In section 2, we consider potential fields and trajectories (and do Kepler's third law as an exercise). In section 3, we prove the special case of the law of conservation of energy mentioned above. In section 4, we do Hamilton's principle. In section 5, we consider the special case of gravity near the earth's surface, while section 6 discusses the general case.

SECTION 1: NO POTENTIAL.

Suppose a small particle of mass m is in space, far from any gravity field or any other forces acting on it. Suppose it moves from point (0, 0, 0) at time t = 0, to point (1, 0, 0) at time t = 1. Suppose its motion is parameterized by curve $\sigma(t) = (x(t), y(t), z(t))$ (for $0 \le t \le 1$). Recall that the kinetic energy of a particle of mass m moving at speed v is $(1/2)mv^2$. At time t, the velocity of our particle is $\sigma'(t) = (x'(t), y'(t), z'(t))$, and so its kinetic energy is $K(t) = (1/2)m||\sigma'(t)||^2$. The expenditure of kinetic energy during the journey is given by the integral $\int_0^1 K(t)dt$.

Goal: Find the curve $\sigma(t)$ that minimizes the expenditure of kinetic energy.

We first claim that expenditure of kinetic energy is minimized when the particle moves in a straight line. For that, let us compare that expenditure over the curve $\sigma(t)$ to the expenditure over the curve $\sigma^{\#}(t)$, where $\sigma^{\#}(t) = (x(t), 0, 0)$. (Note that $\sigma(0) = (x(0), y(0), z(0)) = (0, 0, 0)$, and so x(0) = 0. Thus, $\sigma^{\#}(0) = (x(0), 0, 0) = (0, 0, 0)$. Similarly, since $\sigma(1) = (1, 0, 0)$, we see x(1) = 1, and so $\sigma^{\#}(1) = (1, 0, 0)$. Thus $\sigma^{\#}$ starts at (0, 0, 0) when t = 0, and ends at (1, 0, 0) when t = 1, showing $\sigma^{\#}$ is a possible parameterization of the particle's movement.)

When $\int_0^1 K(t)dt$ is calculated with respect to σ(t), it becomes (1/2)m $\int_0^1 (x'(t))^2 + (y'(t))^2 + (z'(t))^2 dt$. However, when $\int_0^1 K(t)dt$ is calculated over $\sigma^\#$(t), it becomes (1/2)m $\int_0^1 (x'(t))^2 + (0)^2 + (0)^2 dt$, which is obviously equal to or less than the previous integral. Thus the expenditure of kinetic energy over $\sigma^\#$(t) is equal to or less then the expenditure of kinetic energy over σ(t). However, the path of $\sigma^\#$(t) is a straight line, proving the claim.

We now know the path of the parameterized curve which minimizes the expenditure of kinetic energy is a straight line, from (0, 0, 0) to (1, 0, 0). Therefore, we will always be on the x-axis, and so we can ignore the y and z dimensions. We only need the x dimension and time. Therefore, we will only considered 1-dimensional parameterized curves, which have the form x = x(t). We still insist that x(0) = 0, and x(1) = 1. Therefore, we are considering functions x of the variable t, on the interval $0 \le t \le 1$, with the property that x(0) = 0 and x(1) = 1. Some examples are x(t) = t, x(t) = t^2, and x(t) = sin(πt/2). For practice, let us calculate $\int_0^1 K(t)dt$ = (1/2)m $\int_0^1 (x'(t))^2 dt$ for each of these three values of x(t). When x(t) = t, we have x'(t) = 1, and our integral is clearly equal to m/2. When x(t) = t^2, we have x'(t) = 2t, and our integral is (1/2)m $\int_0^1 (2t)^2 dt$ = (1/2)m(4/3) > m/2.

Exercise: Show that when x(t) = sin(πt/2), our integral equals (1/2)m(π^2/8) > m/2. (Use $\cos^2(y)$ = ½ + (cos(2y))/2.)

We see that of our three examples, the expenditure of kinetic energy is minimal when x(t) = t. We now claim that x(t) = t gives the minimum expenditure among all possible (allowable) choices of x(t). We know that when x(t) = t, our integral is m/2. We must show that for any other choice of (allowable) x(t), that integral exceeds m/2. Since our integral equals (1/2)m $\int_0^1 (x'(t))^2 dt$, the next exercise suffices.

Exercise: Let x(t) satisfy x(0) = 0 and x(1) = 1. Let f(t) = x(t) - t. (It is important to note that f(0) = 0 = f(1).) Show $\int_0^1 (x'(t))^2 dt \ge 1$ and equals 1 only when x(t) = t. (Write x(t) = t + f(t).)

We now know the expenditure of kinetic energy is minimized when the particle's motion is described by the parameterized curve σ(t) = (x(t), y(t), z(t)) = (t, 0, 0). The path of that is a straight line, and the velocity vector is σ'(t) = (1, 0, 0), which is a constant, independent of time t. Therefore, we have learned that moving in a straight line at constant speed minimizes expenditure of kinetic energy.

We know that if a particle is in space, far from any gravitational field, or other forces (such as electric or magnetic forces), if you give it a push, it will move in the direction you pushed it at a constant speed. That is its natural motion, (and is called Galileo's first law). What is happening is that the particle moves in such a way as the minimize the expenditure of kinetic energy.

Now let us toss an object into the air. We know from experience that its path is a parabola, and its speed is not constant. Therefore, its motion does not minimize the expenditure of kinetic energy. However, it does minimize something interesting, which we now start to discuss. We will begin by doing the general case, and at the end we will apply what we learn to the special case of tossing an object into the air near the earth's surface.

SECTION 2: POTENTIAL FIELDS, TRAJECTORIES, AND KEPLER'S THIRD LAW.

To save space, we will work in two dimensions, but everything we do works in three dimensions as well.

Notation: Let P(x, y) : $R^2 \to R$ be a function from R^2 to R. (P(x, y) will be called a potential. The reason for that becomes clear in the next section.) Let F(x, y) = -∇P(x, y) = -(∂P/∂x, ∂P/∂y).

We will think of F(x, y) as a force vector at the point (x, y).

Example: Let $P(x, y) = \frac{-GMm}{\sqrt{x^2 + y^2}}$. Calculating shows F(x, y) = -(∂P/∂x, ∂P/∂y) = $-GMm\left(\frac{x}{(x^2 + y^2)^{3/2}}, \frac{y}{(x^2 + y^2)^{3/2}}\right) = \left[\frac{GMm}{x^2 + y^2}\right]\frac{-(x,y)}{\sqrt{x^2 + y^2}}$. That is a scalar times a vector. The vector is obviously a unit vector, pointing in the direction opposite the direction from the origin to the point (x, y). In other words, F(x, y) points from (x, y) towards the origin (just like gravity points from where you are inward towards the center of the earth). As for the scalar, it is a constant GMm divided by $x^2 + y^2$, which is the square of the distance from the origin to (x, y). In other words, the size of the scalar is inversely proportional to the distance from the origin. That is also the way gravity behaves. If we take G to be the universal gravitational constant, M to be the mass of the earth, and m to be (say) your mass, F(x, y) is the force of the earth's gravity pulling on you (in two dimensions). Therefore, gravity is one of the types of force that we will be discussing.

Now let σ(t) = (x(t), y(t)) be a parameterized curve, which we will think of as describing the motion of a small object having mass m.

In what follows, a prime will denote differentiation with respect to time. Thus x′(t) will mean dx(t)/dt. Also, we will sometimes suppress the variable t, in which case the above would be written as x′ = dx/dt.

The velocity and acceleration vector of our object are

$\sigma'(t) = (x'(t), y'(t))$ and $\sigma''(t) = (x''(t), y''(t))$.

At time t, our particle is at position $\sigma(t) = (x(t), y(t))$. At that point, there is a force vector $F(\sigma(t)) = F(x(t), y(t)) = -\nabla P(x(t), y(t))$. We ask if that force vector is the one (and only one) that is causing our particle to move. (Possibly that force is magnetic, and our particle is made of wood, and is not affected by the magnetic force. Or possibly that force is gravitational and does affect our particle, but maybe there are other forces, such as tiny rocket engines also affecting it.)

Newton's law of force tells us the force vector that is causing our particle to move must be $m\sigma''(t)$ (mass times acceleration). Therefore, if $F(\sigma(t))$ is the one and only force affecting our particle, we must have $m\sigma''(t) = F(\sigma(t))$. On the other hand, if $F(\sigma(t)) \neq m\sigma''(t)$, then we know some force besides just $F(x, y)$ is acting on our object.

Definition: $\sigma(t)$ is a trajectory (for the force field $F(x, y)$) if $F(\sigma(t)) = m\sigma''(t)$.

Recalling that $F(\sigma(t)) = -\nabla P(\sigma(t)) = (\frac{-\partial P}{\partial x}(\sigma(t)), \frac{-\partial P}{\partial y}(\sigma(t))$, and $\sigma''(t) = (x''(t), y''(t))$, We see that $\sigma(t)$ is a trajectory for F if and only if $mx''(t) = \frac{-\partial P}{\partial x}(\sigma(t))$, and $my''(t) = \frac{-\partial P}{\partial y}(\sigma(t))$.

Example: With the P(x, y) and F(x, y) as in the previous example, we claim that

$\sigma(t) = (\cos(\sqrt{GM}(t)), \sin(\sqrt{GM}(t))$ is a trajectory. Plugging those values of x and y into the formula $F(x, y) = [\frac{GMm}{x^2 + y^2}]\frac{-(x,y)}{\sqrt{x^2 + y^2}}$ shows $F(\sigma(t)) = -GMm(\cos(\sqrt{GM}(t)), \sin(\sqrt{GM}(t)))$.
Now $\sigma'(t) = (-\sqrt{GM}\sin(\sqrt{GM}(t)), \sqrt{GM}\cos(\sqrt{GM}(t))$ and

$\sigma''(t) = (-GM\cos(\sqrt{GM}(t)), -GM\sin(\sqrt{GM}(t))$, from which we see $m\,\sigma'' = F(\sigma(t))$, which tells us $\sigma(t)$ is a trajectory.

Since any point on $\sigma(t)$ has the form (cos Y, sin Y), all points on that curve are a distance 1 from the origin. In fact, we see the shape of that curve is a unit circle, and as t varies, our particle is traveling around that circle. Recall that the previous example shows our force can be thought of as the force of gravity. Planets travel around the sun in ellipses, but those ellipses are almost circles. Theoretically, there could be a planet which travels around the sun in a circle.

The speed of the particle in the above trajectory is $\|\sigma'(t)\|$, and since we found $\sigma'(t)$ above, we can see that the speed is $\sqrt{GM}$. Notice that is independent of m, the mass of the particle. A light object and a heavy object both in that orbit would travel at the same speed.

Exercise: The parameterized curve $\sigma(t) = (R\cos(Ct), R\sin(Ct))$ gives a circle of radius R. Find the value of the constant C that makes that a trajectory for the force of gravity (used in the previous two examples). Also, find the speed of a particle moving in that trajectory. (The case R = 1 was done in the previous example.)

We digress from our main path to consider Kepler's third law of planetary motion, and how it can be used to learn new facts. We treat this in the following set of exercises.

Exercises: a) (Kepler's third law) Consider the trajectory found in the previous example. Let P (the period) be the amount of time it takes the particle to make one complete revolution. Find P, and show $P^2/R^3 = 4\pi^2/GM$. Note that is independent of R. That fact is Kepler's third law. He discovered it via observation, instead of via mathematics, as we are doing it.

b) Earth is (about) 93,000,000 miles from the sun, and its period is 1 year. Use that to find GM. Mercury is (about) 36,000,000 miles from the sun and its period is (about) 0.24 years. Use that to find GM. Pluto is (about) 3,673,500,000 miles from the sun and its period is about 248 years. Use that to find GM. (All your values of GM should be approximately equal.) Observe that if you knew the value of G, you could find out how much the sun weighs.

c) An artificial satellite and the moon both go around the earth. Suppose the satellite is 700 miles above the earth's surface and takes 2 hours to go around the earth. Use that to find how far away the moon is. (These numbers are approximately accurate).

d) Let m be the mass of the earth, and M be the mass of the sun. Use the information about the artificial satellite in part (c) to find m/M.

Example: Let $P(x, y) = mx^2/2$, so that $F(x, y) = -\nabla P(x, y) = (-mx, 0)$. (This is a very artificial force field, in which all forces are horizontal.) Suppose a particle of mass m starts at the point (2, 1) and has initial velocity (5, ½) at time t = 0. Let us find its trajectory $\sigma(t) = (x(t), y(t))$. We need $m\sigma''(t) = F(\sigma(t))$. Thus, we need $m(x''(t), y''(t)) = (-mx(t), 0)$.

Since there is no vertical component to the force, the vertical component of speed never changes, and so stays at its initial value of ½. Therefore $y'(t) = ½$ for all t, and so $y(t) = 1 + (1/2)t$.

We know $x''(t) = -x(t)$. (This is a differential equation.) We also have the initial conditions that $x(0) = 2$ and $x'(0) = 5$. Using techniques taught in a differential equation course, it is not hard to find that $x(t) = 5\sin(t) + 2\cos(t)$. (Exercise: Verify that.) Thus $\sigma(t) = (5\sin(t) + 2\cos(t), 1 + t/2)$ is the particle's trajectory.

Exercise: Make a sketch of the curve $\sigma(t)$ in the above example for $0 \le t \le 2\pi$. Is it believable that the particle is really moving the way your sketch indicates under the influence of the force in that example?

Exercise: Let $P(x, y) = (m/2)(x^2 + y^2)$. Show that $\sigma(t) = (\sin t, \sin t)$ is a trajectory. Describe the motion of a particle moving along that trajectory (i.e., its orbit), and find its speed as a function of t.

Exercise: Recall that $\cosh t = (1/2)(e^t + e^{-t})$ and $\sinh t = (1/2)(e^t - e^{-t})$. Also recall that $\cosh' t = \sinh t$, $\sinh' t = \cosh t$, and $\cosh^2 t - \sinh^2 t = 1$. (If you have never done so before, verify those facts.) Let $P(x, y) = (-m/2)(x^2 + y^2)$. Show $\sigma(t) = (\cosh t, \sinh t)$ is a trajectory and describe the orbit of a particle moving along that trajectory.

SECTION 3: CONSERVATION OF ENERGY.

Recall that our moving object has kinetic energy equal to 1/2 times m times the square of the speed. That is,

$$K(t) = (m/2)\|\sigma'(t)\|^2 = (m/2)((x'(t)^2 + y'(t)^2).$$

We will also define P(t) to mean $P(\sigma(t))$. Thus

$$P(t) = P(\sigma(t)) = P(x(t), y(t)).$$

Theorem: If $\sigma(t)$ is a trajectory, then $K(t) + P(t)$ is constant as the object travels along $\sigma(t)$.

Proof: It will suffice to show $(K(t) + P(t))' = 0$. Now $(K(t) + P(t))' = K'(t) + P'(t)$, and so it will suffice to show $K'(t) = -P'(t)$.

$$K'(t) = [(m/2)(x'^2 + y'^2)]' = (m/2)(2x'x'' + 2y'y'') = (mx'')x' + (my'')y'.$$

Since $\sigma(t)$ is a trajectory, we know $mx''(t) = \frac{-\partial P}{\partial x}(\sigma(t))$, and $my''(t) = \frac{-\partial P}{\partial y}(\sigma(t))$.

From above, we have $K'(t) = (\frac{-\partial P}{\partial x}(\sigma(t))x' + (\frac{-\partial P}{\partial y}(\sigma(t))y' =$

$$-[(\frac{\partial P}{\partial x}(\sigma(t))\frac{dx}{dt} + (\frac{\partial P}{\partial y}(\sigma(t))\frac{dy}{dt}] = -dP/dt = -P'(t), \text{ (by the chain rule).}$$

The above theorem is a special case of the conservation of energy law of physics. Please ponder the fact that potential energy is a rather abstract concept. You cannot see it. And yet, without the concept, we would not have that very useful law. Of course, Einstein later showed that to have the law work all the time, we had to acknowledge that mass and energy are equivalent--in some sense--and so the law is now the law of conservation of mass and energy. It is one of the great "enlightments" of physics.

SECTION 4: HAMILTON'S PRINCIPLE.

We have a potential energy function P(x, y) and the corresponding force field F(x, y) = $-\nabla$P(x, y).

We will consider a set starting time, which we might as well take to be time 0, and a set ending time which we will take to be time T. Thus, in what follows, time t will always satisfy $0 \le t \le T$.

We will also have a set starting point $P_0 = (x_0, y_0)$ and a set ending point $P_T = (x_T, y_T)$.

We will consider all possible parameterized curves $\sigma(t)$ which our small object of mass m might use to get from point P_0 at time t = 0 to point P_T at time t = T.

Definition: By a permissible curve, we will mean a parameterized curve $\sigma(t)$ such that $\sigma(0) = (x_0, y_0)$ and $\sigma(T) = (x_T, y_T)$. (We will also insist that our curves have continuous first and second derivatives.)

For each permissible $\sigma(t)$, we will consider the integral $I_\sigma = \int (K(t) - P(t))dt$, the integral taken over the curve $\sigma(t)$. HERE, IT IS ALWAY UNDERSTOOD THAT THE LIMITS OF INTEGRATION ARE FROM t = 0 TO t = T. Also, recall that P(t) is a short hand for $P(\sigma(t))$.

We will see that something special is true about that integral when $\sigma(t)$ is a trajectory. (Incidentally, I_σ is called the action integral, and K(t) - P(t) is called the Lagrangian, and is often denoted as L(t).) We will need the following lemma, which is really just the equality of mixed partials, reworked as a result about integration.

Lemma: Let f(t, ε) be a function of both t and ε. Then $\frac{\partial}{\partial\varepsilon}\int_a^b f(t,\varepsilon)dt = \int_a^b \frac{\partial}{\partial\varepsilon} f(t,\varepsilon)dt$.

Proof: Suppose $\frac{\partial}{\partial t}F(t,\varepsilon) = f(t,\varepsilon)$. Then $\frac{\partial}{\partial\varepsilon}\int_a^b f(t,\varepsilon)dt = \frac{\partial}{\partial\varepsilon}(F(b,\varepsilon) - F(a,\varepsilon))$.

We also see that $\frac{\partial}{\partial\varepsilon} f(t,\varepsilon) = \frac{\partial^2}{\partial\varepsilon\partial t}F(t,\varepsilon) = \frac{\partial^2}{\partial t\partial\varepsilon}F(t,\varepsilon)$, and so

$$\int_a^b \frac{\partial}{\partial\varepsilon} f(t,\varepsilon)dt = \int_a^b \frac{\partial^2}{\partial t\partial\varepsilon}F(t,\varepsilon)dt = \frac{\partial}{\partial\varepsilon}F(t,\varepsilon)\,\Big|_a^b = \frac{\partial}{\partial\varepsilon}(F(b,\varepsilon) - F(a,\varepsilon)).$$

NOTATION: Let us fix some permissible $\sigma(t)$ and also consider a second permissible curve $\gamma(t) \neq \sigma(t)$. Let $\delta(t) = \gamma(t) - \sigma(t)$, and for every real number ε, let $\sigma_\varepsilon(t) = \sigma(t) + \varepsilon\delta(t)$. (Note that $\sigma_\varepsilon(t) = \sigma(t)$ when $\varepsilon = 0$ and $\sigma_\varepsilon(t) = \gamma(t)$ when $\varepsilon = 1$. When ε is between 0 and 1, $\sigma_\varepsilon(t)$ will be a curve that can be thought of as being "between" $\sigma(t)$ and $\gamma(t)$.)

Since $\sigma(0) = (x_0, y_0) = \gamma(0)$ and $\sigma(T) = (x_T, y_T) = \gamma(T)$, we see that $\delta(0) = \sigma(0) - \gamma(0) = (0, 0)$, and similarly, $\delta(T) = (0, 0)$.

Since $\sigma(t) \neq \gamma(t)$, we see $\delta(t)$ is not a trivial curve that just sits at one place as time passes. $\delta(t) = (0, 0)$ at times $t = 0$ and $t = T$, but at some other times, $\delta(t) \neq (0, 0)$.

Since $\sigma_\varepsilon(0) = \sigma(0) + \varepsilon\delta(0) = (x_0, y_0) + \varepsilon(0, 0) = (x_0, y_0)$, and similarly $\sigma_\varepsilon(T) = (x_T, y_T)$, we see that $\sigma_\varepsilon(t)$ is a permissible curve. Therefore, we can discuss I_{σ_ε}. For ease of notation, we will let $I(\varepsilon) = I_{\sigma_\varepsilon}$. (Here, $\sigma(t)$ and $\gamma(t)$ are taken as fixed, and so need not be mentioned.)

The integral $I(\varepsilon)$ is a function of the real number ε. Therefore, we can take the derivative $dI(\varepsilon)/d\varepsilon$. In order to take that derivative, we need to understand the integral
$I(\varepsilon) = \int(K(t) - P(t))dt = \int K(t)dt - \int P(t)dt$ (over the curve $\sigma_\varepsilon(t)$).

Let us begin by examining $\int K(t)dt$.

Let $\delta(t) = (x_\delta(t), y_\delta(t))$ and $\sigma_\varepsilon(t) = (x_\varepsilon(t), y_\varepsilon(t))$.

Since $\sigma_\varepsilon(t) = \sigma(t) + \varepsilon\delta(t) = (x(t), y(t)) + \varepsilon(x_\delta(t), y_\delta(t))$, we see

$x_\varepsilon(t) = x(t) + \varepsilon x_\delta(t)$ and $y_\varepsilon(t) = y(t) + \varepsilon y_\delta(t)$.

Note that $x_\varepsilon(t)$ is a function of both t and ε. We can take its partial derivatives with respect to either variable. Per our convention, we continue to use a prime to denote the derivative with respect to t. Thus $x_\varepsilon'(t)$ means $\partial x_\varepsilon(t)/\partial t$

We see that $x_\varepsilon'(t) = x'(t) + \varepsilon x_\delta'(t)$ and $y_\varepsilon'(t) = y'(t) + \varepsilon y_\delta'(t)$.

Since $x_\varepsilon(t) = x(t) + \varepsilon x_\delta(t)$, clearly $\partial x_\varepsilon(t)/\partial\varepsilon = x_\delta(t)$, and similarly $\partial y_\varepsilon(t)/\partial\varepsilon = y_\delta(t)$.

Since $\sigma_\varepsilon(t) = (x_\varepsilon(t), y_\varepsilon(t))$, the velocity of $\sigma_\varepsilon(t)$ is $\sigma_\varepsilon'(t) = (x_\varepsilon'(t), y_\varepsilon'(t))$. Therefore, if our object of mass m is moving along $\sigma_\varepsilon(t)$, then at time t, its kinetic energy is

$K(t) = (m/2)|\sigma_\varepsilon'(t)||^2 =$ (suppressing the t) $(m/2)[(x_\varepsilon')^2 + (y_\varepsilon')^2] =$

$(m/2)[(x' + \varepsilon x_\delta')^2 + (y' + \varepsilon y_\delta')^2] = (m/2)\big[(x'^2 + y'^2) + 2(x'x_\delta' + y'y_\delta')\varepsilon + (x_\delta'^2 + y_\delta'^2)\varepsilon^2\big].$

Therefore, $\int K(t)dt = A\varepsilon^2 + B\varepsilon + C$, where

$A = (m/2) \int x_\delta'^2 + y_\delta'^2 \, dt$ (note A > 0, since $\delta(t)$ does not sit at one place as time passes)

$B = m \int x' x_\delta' + y' y_\delta' \, dt$

$C = (m/2) \int x'^2 + y'^2 \, dt.$

Note that the numbers A, B, and C are independent of ε, and remember that the variable t has been made invisible.

We now see that $I(\varepsilon) = \int (K(t) - P(t))dt = A\varepsilon^2 + B\varepsilon + C - \int P(t)dt$ (the integral over the curve $\sigma_\varepsilon(t)$, as t goes from 0 to T).

Therefore, $\frac{dI(\varepsilon)}{d\varepsilon} = 2A\varepsilon + B - \frac{\partial}{\partial \varepsilon} \int P(t)dt.$

Recalling that P(t) is a short hand for $P(\sigma_\varepsilon(t))$, we see P(t) really depends on both t and ε.

Therefore, using the lemma proved earlier, $\frac{dI(\varepsilon)}{d\varepsilon} = 2A\varepsilon + B - \int \frac{\partial}{\partial \varepsilon} P(\sigma_\varepsilon(t)dt =$

$2A\varepsilon + B - \int \frac{\partial}{\partial \varepsilon} P(x_\varepsilon(t), y_\varepsilon(t))dt =$ (by the Chain Rule)

$2A\varepsilon + B - \int [(\frac{\partial P}{\partial x_\varepsilon})(\frac{\partial x_\varepsilon}{\partial \varepsilon}) + (\frac{\partial P}{\partial y_\varepsilon})(\frac{\partial y_\varepsilon}{\partial \varepsilon})]dt =$

(from above) $2A\varepsilon + B - \int [x_\delta(t)(\frac{\partial P}{\partial x_\varepsilon}) + y_\delta(t)(\frac{\partial P}{\partial y_\varepsilon})]dt.$

The above is our formula for $\frac{dI(\varepsilon)}{d\varepsilon}$. We will now evaluate that at $\varepsilon = 0$. That is, we will find $\frac{dI(\varepsilon)}{d\varepsilon}|_0$.

Since when $\varepsilon = 0$, $2A\varepsilon + B$ is just B, we get

$$\frac{dI(\varepsilon)}{d\varepsilon}|_0 = B - \int [x_\delta(t)(\frac{\partial P}{\partial x_\varepsilon})|_0 + y_\delta(t)(\frac{\partial P}{\partial y_\varepsilon})|_0]dt.$$

Since $x_\varepsilon(t) = x(t) + \varepsilon x_\delta(t)$, we see $x_0(t) = x(t)$, and so

$\frac{\partial P}{\partial x_\varepsilon}|_0 = \frac{\partial P}{\partial x}$, and similarly, $\frac{\partial P}{\partial y_\varepsilon}|_0 = \frac{\partial P}{\partial y}$.

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Example: Suppose $P(x, y) = x^3y^2$, so $\frac{\partial P}{\partial x} = 3x^2y^2$.

Now $P(x_\varepsilon, y_\varepsilon) = x_\varepsilon{}^3y_\varepsilon{}^2$, and $\frac{\partial P}{\partial x_\varepsilon} = 3x_\varepsilon{}^2y_\varepsilon{}^2$.

Thus $\frac{\partial P}{\partial x_\varepsilon}|_0 = 3x_\varepsilon{}^2y_\varepsilon{}^2|_0 = 3x_0{}^2y_0{}^2 = 3x^2y^2 = \frac{\partial P}{\partial x}$ (since $x_\varepsilon = x + \varepsilon x_\delta$ is x when $\varepsilon = 0$).

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We conclude that

$\frac{dI(\varepsilon)}{d\varepsilon}|_0 = B - \int [x_\delta(t)(\frac{\partial P}{\partial x}) + y_\delta(t)(\frac{\partial P}{\partial y})]dt$, the integral over the curve $\sigma(t)$ (= $\sigma_\varepsilon(t)$ when $\varepsilon = 0$), and the limits from t = 0 to t = T, as always.

HAMILTON'S PRINCIPLE: $\frac{dI(\varepsilon)}{d\varepsilon}|_0 = 0$ IFF $\sigma(t)$ is a trajectory.

Proof: First, suppose $\sigma(t)$ is a trajectory. We will show $\frac{dI(\varepsilon)}{d\varepsilon}|_0 = 0$ (regardless of what the curve $\gamma(t)$ is, just so long as it is a permissible curve).

Since $\sigma(t) = (x(t), y(t))$ is a trajectory, we know from earlier that

$mx''(t) = \frac{-\partial P}{\partial x}(\sigma(t))$, and $my''(t) = \frac{-\partial P}{\partial y}(\sigma(t))$.

We know $\frac{dI(\varepsilon)}{d\varepsilon}|_0 = B - \int [x_\delta(t)(\frac{\partial P}{\partial x}) + y_\delta(t)(\frac{\partial P}{\partial y})]dt.$

Now $\frac{\partial P}{\partial x}$ is evaluated at σ(t) = (x(t), y(t)), and so $\frac{\partial P}{\partial x}$ really means $\frac{\partial P}{\partial x}$(σ(t)), which we just saw equals -mx″(t) (since σ(t) is a trajectory). Similarly, $\frac{\partial P}{\partial y}$ = -my″(t). Therefore,

$$\frac{dI(\varepsilon)}{d\varepsilon}|_0 = B + m\int x_\delta(t)x''(t) + y_\delta(t)y''(t)\,dt = B + m\int x_\delta x'' + y_\delta y''\,dt \text{ (surpressing the t).}$$

Recall that $B = m\int x'x_\delta' + y'y_\delta'\,dt$.

Consider $\int x'(t)x_\delta'(t)\,dt$ and recall this integral goes from t = 0 to t = T. We evaluate it, using integration by parts.

Let u = x′ and dv = x_δ'dt. Then du = x″dt and v = x_δ. Thus $\int x'x_\delta't = x'x_\delta|_0^T - \int x_\delta x''dt$.

However, δ(0) = (0, 0) = δ(T), so that $x_\delta(0) = 0 = x_\delta(T)$, and so

$x'x_\delta|_0^T = x'(T)x_\delta(T) - x'(0)x_\delta(0) = 0$. Therefore, $\int x'x_\delta dt = -\int x_\delta x''dt$.

Similarly, $\int y'y_\delta'\delta t = -\int y_\delta y''dt$. Therefore, $B = m\int x'x_\delta' + y'y_\delta'\,dt =$

$-m\int x_\delta x'' + y_\delta y''\,dt$. We now see $\frac{dI(\varepsilon)}{d\varepsilon}|_0 = B + m\int x_\delta x'' + y_\delta y''\,dt = 0$, as desired. This completes one direction of the argument.

For the converse, we assume that for any permissible γ(t) and δ(t) = γ(t) - σ(t), that $\frac{dI(\varepsilon)}{d\varepsilon}|_0 = 0$. We must show σ(t) is a trajectory (for F(x, y)). That is, we need

$$mx''(t) = -(\frac{\partial P}{\partial x})(\sigma(t)) \text{ and } my''(t) = -(\frac{\partial P}{\partial y})(\sigma(t)).$$

We will show the first of these two equations is true, the other being similar.

We need $-mx''(t) - (\frac{\partial P}{\partial x})(\sigma(t)) = 0$ for t satisfying 0 ≤ t ≤ T. Suppose that fails to be true for t = t# with 0 ≤ t# ≤ T. (We will derive a contradiction.)

Since we are assuming $-mx''(t\#) - \frac{\partial P}{\partial x}(\sigma(t\#))$ is not zero, it is either positive or negative. Let us assume it is positive (the other case being similar).

Since $-mx''(t\#) - \frac{\partial P}{\partial x}(\sigma(t\#)) > 0$, by continuity the same is true for all t sufficiently close to t#. Therefore, if t# happens to be 0, we can replace it with another choice of t# with t# > 0. Similarly, if t# = T, we replace it with another t# with t# < T. Thus, we may assume 0 < t# < T.

Again by continuity, there is an interval (call it J) containing t# in its interior on which $-mx''(t) - \frac{\partial P}{\partial x}(\sigma(t))$ is always positive. Making J smaller if necessary, we may assume 0 and T are not in J.

It is easy to see there is a function $x_\delta(t)$ which equals 0 at all t outside of J, is never negative, and is positive at some points in J. (Since 0 and T are not in J, $x_\delta(0) = 0 = x_\delta(T)$.)

We claim $x_\delta(t)[-mx''(t) - (\frac{\partial P}{\partial x})(\sigma(t))]$ is never negative, and sometimes positive (for $0 \le t \le T$). It is sometimes positive since $-mx''(t) - (\frac{\partial P}{\partial x})(\sigma(t))$ is positive for all t in J and $x_\delta(t)$ is positive for some t in J. Also, since $x_\delta(t)$ is never negative, to have $x_\delta(t)[-mx''(t) - (\frac{\partial P}{\partial x})(\sigma(t))]$ be negative, we would need $x_\delta(t) \neq 0$ and $-mx''(t) - (\frac{\partial P}{\partial x})(\sigma(t)) < 0$. However, that situation does not apply to t outside of J, nor to t inside of J.

The claim shows, $\int x_\delta(t)[-mx''(t) - (\frac{\partial P}{\partial x})(\sigma(t))]\, dt > 0$ (since the integral is over the interval $0 \le t \le T$).

Let $\delta(t) = (x_\delta(t), 0)$, and note that $\delta(t)$ is (0, 0) when t = 0 and when t = T.

Let $\gamma(t) = \delta(t) + \sigma(t)$, and note that $\gamma(0) = \delta(0) + \sigma(0) = (0, 0) + (x_0, y_0) = (x_0, y_0)$, and similarly, $\gamma(T) = (x_T, y_T)$, so that $\gamma(t)$ is one of our permissible curves. Also, since $\delta(t)$ is sometimes nonzero, the curves $\gamma(t)$ and $\sigma(t)$ are not equal.

As before, we define $\sigma_\varepsilon(t) = \sigma(t) + \varepsilon\delta(t)$ for this $\delta(t)$. We also define the integral $I(\varepsilon)$ as before. Recall that our assumption is that $\frac{dI(\varepsilon)}{d\varepsilon}|_0 = 0$.

We earlier saw $\frac{dI(\varepsilon)}{d\varepsilon}|_0 = B - \int [x_\delta(t)(\frac{\partial P}{\partial x}) + y_\delta(t)(\frac{\partial P}{\partial y})]dt$.

Since $\delta(t) = (x_\delta(t), 0)$, we have $y_\delta(t) = 0$ for all t.

Thus $\frac{dI(\varepsilon)}{d\varepsilon}|_0 = B - \int x_\delta(t)(\frac{\partial P}{\partial x})dt.$

We also previously saw that

$B = -m\int x_\delta(t)x''(t) + y_\delta(t)y''(t)dt = -m\int x_\delta(t)x''(t)dt$ (since $y_\delta(t) = 0$).

Therefore, $\frac{dI(\varepsilon)}{d\varepsilon}|_0 = B - \int x_\delta(t)(\frac{\partial P}{\partial x})\, dt = -m\int x_\delta(t)x''(t)\, dt - \int x_\delta(t)(\frac{\partial P}{\partial x})\, dt =$

$\int x_\delta(t)[-mx''(t) - \frac{\partial P}{\partial x}(\sigma(t))]\, dt > 0$ (from above).

This contradicts that $\frac{dI(\varepsilon)}{d\varepsilon}|_0$ was assumed to be 0, and completes the proof.

Hamilton's principle says that if $\sigma(t)$ is a trajectory, then $\frac{dI(\varepsilon)}{d\varepsilon}|_0 = 0$ (for any permissible $\gamma(t)$), which means that $\varepsilon = 0$ is a critical point for $I(\varepsilon)$. Thus $\varepsilon = 0$ might be a local maximum or local minimum (or point of inflection). Note that for one choice of $\gamma(t)$ it might be a minimum, while for a different $\gamma(t)$ it might be a maximum. That is similar to the idea of a 'saddle point', which gives a local maximum when approached from one direction, but a local minimum when approached from different direction. The next example and exercise illustrate possibilities.

Example: Let C be a constant, and let $P(x, y) = Cmy^2$, so that $F(x, y) = -\nabla P(x, y) = (0, -2Cmy)$. In this example, we will let $T = 1$, so we have $0 \leq t \leq 1$. Note that at the origin, $F(0, 0) = (0, 0)$, showing there is no force at the origin. Thus, an object that starts at rest at the origin will simply stay at the origin. Therefore, $\sigma(t) = (0, 0)$ is a trajectory. Now consider $\gamma(t) = (0, t^2 - t)$. We see $\gamma(0) = (0, 0) = \gamma(1)$, showing γ is permissible. We have $\delta(t) = \gamma(t) - \sigma(t) = \gamma(t) = (0, t^2 - t)$, and so $\delta_\varepsilon(t) = \sigma(t) + \varepsilon\delta(t) = (0, \varepsilon(t^2 - t))$. Thus $\delta_\varepsilon'(t) = (0, \varepsilon(2t - 1))$, and $K(t) = (1/2)m\|\delta_\varepsilon'(t)\|^2 = (m/2)\varepsilon^2(2t - 1)^2$. When we integrate $K(t)$ over δ_ε (the limits of integration being 0 to 1), we get $\int K(t)dt = m\varepsilon^2/6$. Now $P(t) = P(\delta_\varepsilon(t)) = Cm(y_\varepsilon(t))^2 = Cm\varepsilon^2(t^2 - t)^2$. Therefore, $\int P(t)dt = Cm\varepsilon^2/30$. Therefore, we have $I(\varepsilon) = (K(t) - P(t))dt = m(1/6 - C/30)\varepsilon^2$.

$I(\varepsilon)$ is just a parabola. If $C < 5$, then $m(1/6 - C/30) > 0$, and the parabola opens upward, giving a minimum at $\varepsilon = 0$. If $C > 5$, the parabola opens downward, and there is a maximum at $\varepsilon = 0$. If $C = 5$, $I(\varepsilon)$ is constantly 0.

Exercise: In the above example, change $\gamma(t)$ to now be $\gamma(t) = (t^2 - t, 0)$. Show $I(\varepsilon) = m\varepsilon^2/6$ regardless of the value of C, showing that $I(\varepsilon)$ has a minimum at $\varepsilon = 0$. (Therefore, for $C > 5$, whether $I(\varepsilon)$ has a maximum of minimum at $\varepsilon = 0$ depends on the choice of $\gamma(t)$.)

We next look at a particularly easy case.

SECTION 5: GRAVITY NEAR THE EARTH'S SURFACE.

The acceleration due to gravity near the earth's surface is usually denoted g. That means an object of mass m is pulled down by a force equal to mg. This is (approximately) independent of its position (x, y), so long as y (the height above the surface) is reasonably small. Therefore, $F(x, y) = (0, -mg)$. We easily see $F = -\nabla P$, where $P(x, y) = gmy$.

Let $\sigma(t)$ $(0 \leq t \leq T)$ be the trajectory of (say) a baseball. Let $\gamma(t)$ be any other permissible curve (so that $\gamma(0) = \sigma(0)$ and $\gamma(T) = \sigma(T)$). Here, let us also assume the path described by $\gamma(t)$ never gets too high above the earth, nor travels so far horizontally that the earth's curvature becomes noticeable.

We claim that if $\gamma(t) \neq \sigma(t)$, then $\int K(t) - P(t)dt$ calculated over $\sigma(t)$ is smaller than that integral calculated over $\gamma(t)$. Thus, taken over $\sigma(t)$, our integral is an absolute minimum (not just a local minimum).

We prove our claim. Using our standard notation, we already know that taken over $\delta_\varepsilon(t)$, we have $\int K(t)dt = A\varepsilon^2 + B\varepsilon + C$. Now $\int P(t)dt = \int gmy_\varepsilon(t) = gm\int y(t) + \varepsilon y_\delta(t)dt = gm\int y(t)dt + \varepsilon gm\int y_\delta(t)dt$.

Therefore, $I(\varepsilon) = \int K(t) - P(t)dt = A\varepsilon^2 + (B - gm\int y_\delta(t)dt)\varepsilon + (C - gm\int y(t)dt)$. That is just a parabola in the variable ε. Since we know $A > 0$, this parabola opens upward, and so has a unique critical point, at its minimum. Since $\sigma(t)$ is the ball's trajectory, Hamilton's principle tells us that $\varepsilon = 0$ is a critical point for $I(\varepsilon)$. Therefore, the minimum value of $I(\varepsilon)$ occurs at $\varepsilon = 0$. However, when $\varepsilon = 0$, $\sigma_\varepsilon(t) = \sigma(t)$.

For gravity near the earth's surface, things are particularly simple because the potential function $P(x, y) = gmy$ is so very simple. Another very easy case is when $P(x, y) = 0$. That is the case that there is no potential energy, which was treated in section 1.

In general, the situation is more complicated. However, it can be dealt with when T is sufficiently small, as we see next.

SECTION 6: THE GENERAL CASE, WHEN T IS SMALL.

The example at the end of section 4 shows that if $\sigma(t)$ is a trajectory (and $\gamma(t)$ is permissible), then $I(\varepsilon)$ can have a a local maximum or local minimum at $\varepsilon = 0$. We will now show that if T is sufficiently small, then we will in fact have a local minimum. We do that via the second derivative test, by showing if T is sufficiently small, then the second derivative of $I(\varepsilon)$ with respect to ε will always be positive at $\varepsilon = 0$, (even if $\sigma(t)$ is not a trajectory).

We already know $\frac{dI(\varepsilon)}{d\varepsilon} = 2A\varepsilon + B - \int [x_\delta(t)(\frac{\partial P}{\partial x_\varepsilon}) + y_\delta(t)(\frac{\partial P}{\partial y_\varepsilon})]dt.$

When we differentiate this with respect to ε, (and use the lemma from earlier), we get

$$\frac{d^2I(\varepsilon)}{d\varepsilon^2} = 2A - \frac{d}{d\varepsilon}\int[x_\delta(t)(\frac{\partial P}{\partial x_\varepsilon}) + y_\delta(t)(\frac{\partial P}{\partial y_\varepsilon})]dt = 2A - \int\frac{d}{d\varepsilon}[x_\delta(t)(\frac{\partial P}{\partial x_\varepsilon}) + y_\delta(t)(\frac{\partial P}{\partial y_\varepsilon})]dt.$$

Let us calculate $\frac{d}{d\varepsilon}[x_\delta(t)(\frac{\partial P}{\partial x_\varepsilon})]$. Since $x_\delta(t)$ does not depend upon ε, that first becomes $x_\delta(t)\frac{d}{d\varepsilon}(\frac{\partial P}{\partial x_\varepsilon})$. Now using the chain rule,

$$\frac{d}{d\varepsilon}(\frac{\partial P}{\partial x_\varepsilon}) = [\frac{\partial(\partial P/\partial x_\varepsilon)}{\partial x_\varepsilon}]\frac{\partial x_\varepsilon}{\partial \varepsilon} + [\frac{\partial(\partial P/\partial x_\varepsilon)}{\partial y_\varepsilon}]\frac{\partial y_\varepsilon}{\partial \varepsilon}.$$

Recall that $\frac{\partial x_\varepsilon(t)}{\partial\varepsilon} = x_\delta(t)$ and $\frac{\partial y_\varepsilon(t)}{\partial\varepsilon} = y_\delta(t)$.

Also the shorthand for $\frac{\partial(\partial P/\partial x_\varepsilon)}{\partial x_\varepsilon}$ is $\frac{\partial^2 P}{\partial x_\varepsilon{}^2}$, and similarly the shorthand for $\frac{\partial(\partial P/\partial x_\varepsilon)}{\partial y_\varepsilon}$ is $\frac{\partial^2 P}{\partial y_\varepsilon \partial x_\varepsilon}$. Therefore, $\frac{d}{d\varepsilon}[x_\delta(t)(\frac{\partial P}{\partial x_\varepsilon})] = x_\delta(t)[x_\delta(t)\frac{\partial^2 P}{\partial x_\varepsilon{}^2} + y_\delta(t)\frac{\partial^2 P}{\partial y_\varepsilon\partial x_\varepsilon}]$.

Similarly, we see that $\frac{d}{d\varepsilon}[y_\delta(t)(\frac{\partial P}{\partial y_\varepsilon})] = y_\delta(t)[x_\delta(t)\frac{\partial^2 P}{\partial x_\varepsilon\partial y_\varepsilon} + y_\delta(t)\frac{\partial^2 P}{\partial y_\varepsilon{}^2}]$.

However, by the equality of mixed partials, we also know $\frac{\partial^2 P}{\partial y_\varepsilon\partial x_\varepsilon} = \frac{\partial^2 P}{\partial x_\varepsilon\partial y_\varepsilon}$.

We earlier had $\frac{d^2I(\varepsilon)}{d\varepsilon^2} = 2A - \int\frac{d}{d\varepsilon}[x_\delta(t)(\frac{\partial P}{\partial x_\varepsilon}) + y_\delta(t)(\frac{\partial P}{\partial y_\varepsilon})]dt.$

Combining that with the above facts gives

$$\frac{d^2I(\varepsilon)}{d\varepsilon^2} = 2A - \int \quad (x_\delta(t))^2 \frac{\partial^2 P}{\partial x_\varepsilon{}^2} + 2x_\delta(t)y_\delta(t)\frac{\partial^2 P}{\partial x_\varepsilon \partial y_\varepsilon} + (y_\delta(t))^2 \frac{\partial^2 P}{\partial y_\varepsilon{}^2} \text{ dt.}$$

Recall that we earlier showed $\frac{\partial P}{\partial x_\varepsilon}|_0 = \frac{\partial P}{\partial x}$. Applying similar reasoning to our second derivatives shows that

$$\frac{d^2I(\varepsilon)}{d\varepsilon^2}|_0 = 2A - \int \quad (x_\delta(t))^2 \frac{\partial^2 P}{\partial x^2} + 2x_\delta(t)y_\delta(t)\frac{\partial^2 P}{\partial x \partial y} + (y_\delta(t))^2 \frac{\partial^2 P}{\partial y^2} \text{ dt.}$$

We will show that if T is sufficiently small, then $\frac{d^2I(\varepsilon)}{d\varepsilon^2}|_0$ is positive. This is true even if σ(t) is not a trajectory! In the case that σ(t) is a trajectory, we will have that the first derivative of I(ε) is 0 at ε = 0, and also that the second derivative is positive when ε = 0. That will tell us that I(ε) has a local minimum at ε = 0 (when σ(t) is a trajectory and T is sufficiently small). That will be true regardless of the choice of permissible γ(t) we use, and so we will be able to say that ∫K(t) – P(t)dt (thought of as a function of the permissible γ(t) over which it is calculated) has a local minimum at σ(t). (Perhaps it is not an absolute minimum.)

Using the above formula for $\frac{d^2I(\varepsilon)}{d\varepsilon^2}|_0$, we wish to show it is positive for T sufficiently small. Recall, we earlier saw that A > 0, and so 2A > 0. One might therefore think that for very small T, the integral in the formula will be very small and so 2A minus that integral will be positive. Regrettably, it is not that easy. Remember that $A = (m/2)\int x_\delta'^2 + y_\delta'^2$ dt, and this integral also is for $0 \le t \le T$. Thus, as T gets small, so does A. To surmount that problem, we introduce some interesting results, starting with a version of the Cauchy-Schwartz Inequality.

Lemma: Let a < b, and let f(t), g(t) be two functions.
Then $\left|\int_a^b f(t)g(t)dt\right| \le \sqrt{\left(\int_a^b f^2(t)dt\right)\left(\int_a^b g^2(t)dt\right)}$.

Proof: If f(t) is constantly 0 for $a \le t \le b$, then the result is obviously true, as both sides equal 0. Thus we assume f(t) is not constantly 0 on that interval, so that $\int_a^b f^2(t)dt \neq 0$, allowing us to (later) divide by it. Let z be any number. Then $0 \le \int_a^b (zf(t) - g(t))^2 dt =$

$\int_a^b z^2f^2(t) - 2zf(t)g(t) + g^2(t)\, dt = z^2 \int_a^b f^2(t)dt - 2z \int_a^b f(t)g(t)dt + \int_a^b g^2(t)dt.$

Now let $z = \frac{\int_a^b f(t)g(t)dt}{\int_a^b f^2(t)dt}$. As we know the resulting expression is equal to or greater than 0, the lemma follows by easy algebra (and the fact that $\int_a^b h^2(t)dt \geq 0$ for any h(t)).
(Exercise: verify that the value of z we used was the one that minimizes the above quadratic in z.)

Corollary: Let $a \leq b$, and suppose f(t) is a function. Then $|f(b) - f(a)| \leq \sqrt{(b-a)\int_a^b (f'(t))^2 dt}$.

Proof: The case a = b is trivial, since then $|f(b) - f(a)| = 0 = b - a$. Assume $a < b$.
Using the above Cauchy-Schwartz inequality, (with g(t) = 1 and with f'(t) instead of f(t)), we get

$$|f(b) - f(a)| = |\int_a^b 1\, f'(t)dt| \leq \sqrt{(\int_a^b 1^2 dt)(\int_a^b (f'(t))^2 dt)} = \sqrt{(b-a)(\int_a^b (f'(t))^2 dt)}.$$

Corollary: Let $a < b$, and suppose f(t) is a function and f(a) = 0. Then for any s with $a \leq s \leq b$, we have $|f(s)| \leq \sqrt{(b-a)(\int_a^b (f'(t))^2 dt}$.

Proof: By the previous corollary applied to $a \leq s$, we have

$|f(s)| = |f(s) - f(a)| \leq \sqrt{(s-a)(\int_a^s (f'(t))^2 dt)}$. However, this last is equal to or less than $\sqrt{(b-a)(\int_a^b (f'(t))^2 dt)}$ (since $b - a \geq s - a$ and $\int_a^b (f'(t))^2 dt \geq \int_a^s (f'(t))^2 dt$.

Corollary: Let $a < b$ and let f(t) be a function with f(a) = 0.

Then $\int_a^b f^2(t)dt \leq (b - a)^2 \int_a^b (f'(t))^2 dt$.

Proof: By the previous corollary, if $a \le s \le b$, then $f^2(s) = |f(s)|^2 \le (b - a) \int_a^b (f'(t))^2 dt$. This last expression is a constant, which we call C, so that $f^2(s) \le C$, for all s with $a \le s \le b$. That is, $f^2(t) \le C$ for all $a \le t \le b$. Therefore, $\int_a^b f^2(t)dt \le \int_a^b C dt = (b - a)C = (b - a)^2 \int_a^b (f'(t))^2 dt$.

Exercise: Show that if $f(0) = 0$, then $\int_0^1 (f'(x))^2 - (f(x))^2 dx \ge 0$.
Verify that fact directly for $f(x) = \sin x$. (Note that $(\sin'(1))^2 - (\sin(1))^2 < 0$. Therefore, the integrand is not always positive, but the integral is positive.)

Lemma: For numbers c and d, $(|c| + |d|)^2 \le 2(c^2 + d^2)$.

Proof: $0 \le (|c| - |d|)^2 = |c|^2 - 2|c||d| + |d|^2$, so $2|c||d| \le |c|^2 + |d|^2$.

Thus $(|c| + |d|)^2 = |c|^2 + 2|c||d| + |d|^2 \le 2(|c|^2 + |d|^2) = 2(c^2 + d^2)$.

We earlier said we will show that when T is sufficiently small, $\frac{d^2 I(\varepsilon)}{d\varepsilon^2}|_0$ will be positive.

We are now ready to prove that. The hypothesis of the next theorem shows what we mean by 'sufficiently small'.

Theorem: Suppose H is a positive number equal to or greater than all of $|\frac{\partial^2 P}{\partial x^2}(\sigma(t))|$, $|\frac{\partial^2 P}{\partial x \partial y}(\sigma(t))|$, and $|\frac{\partial^2 P}{\partial y^2}(\sigma(t))|$ for all t with $0 \le t \le T$. Suppose also that $T < \sqrt{m/2H}$. Then for all permissible $\gamma(t)$ and $\delta(t) = \gamma(t) - \sigma(t)$, $\frac{d^2 I(\varepsilon)}{d\varepsilon^2}|_0 > 0$.
(Here, $\sigma(t)$ need not be a trajectory.)

Remark: Suppose $T \ge \sqrt{m/2H}$. Then take a smaller T. That is, consider a shorter part of the trajectory. On that shorter trajectory, the value of H will not get larger, and might get smaller. If it does get smaller, $\sqrt{m/2H}$ gets bigger, making $T < \sqrt{m/2H}$ more likely. For sufficiently small T, that inequality will hold.

Proof: We previously saw (suppressing the t in $x_\delta(t) = x_\delta$ and $y_\delta(t) = y_\delta$), that

$$\frac{d^2 I(\varepsilon)}{d\varepsilon^2}|_0 = 2A - \int (x_\delta)^2 \frac{\partial^2 P}{\partial x^2} + 2x_\delta y_\delta \frac{\partial^2 P}{\partial x \partial y} + (y_\delta)^2 \frac{\partial^2 P}{\partial y^2} \, dt.$$

(Recall this integral is for $0 \le t \le T$, with $T < \sqrt{m/2H}$.)

Consider the above integral. If we replace each term in the integrand with its absolute value, we have made the integrand equal or bigger, and so have made the integral equal or bigger. Since we are subtracting the integral from 2A, those absolute values make the result equal or smaller. Thus,

$$\frac{d^2I(\varepsilon)}{d\varepsilon^2}|_0 \geq 2A - \int |x_\delta|^2 \left|\frac{\partial^2 P}{\partial x^2}\right| + 2|x_\delta||y_\delta|\left|\frac{\partial^2 P}{\partial x \partial y}\right| + |y_\delta|^2 \left|\frac{\partial^2 P}{\partial y^2}\right| \, dt.$$

As each second partial in the above integral is evaluated at the various $\sigma(t)$ for $0 \leq t \leq T$, we know the absolute value of each of those second partials is at most H. Thus, replacing the absolute values of those second partials with H makes the integral equal or bigger and so makes the result equal or smaller. Therefore,

$$\frac{d^2I(\varepsilon)}{d\varepsilon^2}|_0 \geq 2A - \int |x_\delta|^2 H + 2|x_\delta||y_\delta|H + |y_\delta|^2 H dt =$$

$2A - H\int |x_\delta|^2 + 2|x_\delta||y_\delta| + |y_\delta|^2 dt = 2A - H\int (|x_\delta| + |y_\delta|)^2 dt \geq$ (using the previous lemma)

$2A - 2H\int x_\delta^2 + y_\delta^2 dt =$ (using the earlier value of A)

$m\int x_\delta'^2 + y_\delta'^2 \, dt - 2H\int x_\delta^2 + y_\delta^2 dt =$

$m\left[\int x_\delta'^2 - (2H/m)x_\delta^2 dt + \int y_\delta'^2 - (2H/m)y_\delta^2 dt\right].$

We wish to show this is positive. It will suffice to show at least one of the two integrals in this last expression is positive, and the other is nonnegative.

Consider $\int x_\delta'^2 - (2H/m)x_\delta^2 dt$. Since $\delta(0) = (0, 0)$, we have $x_\delta(0) = 0$. By the last of the above corollaries (with $a < b$ being $0 < T$), we have

$\int x_\delta^2 dt \leq T^2 \int x_\delta'^2 dt \leq (m/2H) \int x_\delta'^2 dt$ (since $T < \sqrt{m/2H}$).

Note that the strict inequality in $T < \sqrt{m/2H}$ does not quite allow us to have a strict inequality in

$T^2 \int x_\delta'^2 dt \leq (m/2H) \int x_\delta'^2 dt$. The reason is that perhaps $\int x_\delta'^2 dt = 0$, in which case both sides of our inequality are 0. Now $\int x_\delta'^2 dt = 0$ IFF $x_\delta'(t) = 0$ for all $0 \leq t \leq T$.

Thus, we see that $\int x_\delta^2 dt \leq (m/2H)\int x_\delta'^2 dt$, and the inequality is strict if $x_\delta'(t)$ is not always 0.

Equivalently, $(2H/m)\int x_\delta^2 dt \le \int x_\delta'^2 dt$, and the inequality is strict if $x_\delta'(t)$ is not always 0.

Therefore, $\int x_\delta'^2 - (2H/m)x_\delta^2 dt \ge 0$, and the inequality is strict if $x_\delta'(t)$ is not always 0.

Similarly, $\int y_\delta'^2 - (2H/m)y_\delta^2 dt \ge 0$, and the inequality is strict if $y_\delta'(t)$ is not always 0.

Recall we had $\frac{d^2 I(\varepsilon)}{d\varepsilon^2}|_0 \ge m\left[\int x_\delta'^2 - (2H/m)x_\delta^2 dt + \int y_\delta'^2 - (2H/m)y_\delta^2 dt\right]$.

We want this last to be positive. The above comments show it is nonnegative, (since each of the two integrals is nonnegative), and will be positive if at least one of x_δ' or y_δ' is not always 0 for $0 \le t \le T$. We claim that is the case. Suppose to the contrary that $x_\delta'(t) = 0 = y_\delta'(t)$ for $0 \le t \le T$. Then $\delta'(t) = (0, 0)$ for $0 \le t \le T$. That says the velocity vector for $\delta(t)$ is the zero vector, meaning $\delta(t)$ is just sitting at one point. Since $\delta(0) = (0, 0)$, we must have $\delta(t) = (0, 0)$ for all t with $0 \le t \le T$. So $(0, 0) = \delta(t) = \gamma(t) - \sigma(t)$, showing $\sigma(t) = \gamma(t)$ for all $0 \le t \le T$. That violates our early assumption that $\sigma(t)$ and $\gamma(t)$ are different curves.Therefore, one of x_δ' or y_δ' is not constantly 0, and so $\frac{d^2 I(\varepsilon)}{d\varepsilon^2}|_0 \ge 0$.

Examples: a) Recall in an earlier example, we had $P(x, y) = Cmy^2$. We saw that $\sigma(t) = (0, 0)$ was a trajectory (with $0 \le t \le 1 = T$). We let $\gamma(t) = (0, t^2 - t)$, and found that $I(\varepsilon) = m(1/6 - C/30)\varepsilon^2$.

Let us find the H of the previous theorem for this P(x, y). We calculate that $\left|\frac{\partial^2 P}{\partial x^2}(x, y)\right| = 0$, $\left|\frac{\partial^2 P}{\partial x \partial y}(x, y)\right| = 0$, and $\left|\frac{\partial^2 P}{\partial y^2}(x, y)\right| = |2Cm|$. Therefore, we may take $H = |2Cm|$ (for all points (x, y) in the plane.) The theorem says if $T < \sqrt{\frac{m}{2H}} = \frac{1}{2\sqrt{|C|}}$, then $I(\varepsilon)$ has a local minimum at $\varepsilon = 0$. In that example, we had $T = 1$, and so the above inequality holds whenever $|C| < ¼$. That is, the theorem tells us $I(\varepsilon)$ has a local minimum at $\varepsilon = 0$ if $|C| \le ¼$. (Of course, we already knew from that earlier example that $I(\varepsilon)$ has local minimum at $\varepsilon = 0$ if $C < 5$, which is stronger than what the theorem tells us.)

b) Now let us take that earlier example, and specify that $C = 6$. Then $I(\varepsilon) = m(1/6 - C/30)\varepsilon^2 = (-m/30)\varepsilon^2$. Thus $I(\varepsilon)$ has a maximum at $\varepsilon = 0$. (Note that $I(0) = 0$ but $I(1) = -m/30$.) Recall, $T = 1$. The above theorem assures us that it T is small enough, we will get a local minimum, not a local maximum, and therefore we conclude that $T = 1$ is too big. We ask how small T must be to get a local minimum? As in part (a), we need $T < \frac{1}{2\sqrt{|C|}} = \frac{1}{\sqrt{24}}$ (for $C = 6$). In the next exercise, we take $T = 1/10 < \frac{1}{\sqrt{24}}$.

Exercise: With C = 6 (as above), consider $0 \le t \le 1/10 = T$. As before, $\sigma(t) = (0, 0)$ is a trajectory. Let $\gamma^*(t) = (0, 100t^2 - 10t)$. Note that $\gamma^*(0) = (0, 0) = \gamma^*(1/10)$, showing $\gamma^*(t)$ is permissible. Show (via calculation rather than via the theorem) that with that choice of $\gamma^*(t)$, $I(\varepsilon)$ has a local minimum at $\varepsilon = 0$.

Exercise: We refer to part (b) of the previous example, (so C = 6), and take $\gamma^*(t)$ as in the previous exercise. Consider a particle of mass m moving along the curve, $\gamma^*(t)$. It starts at (0, 0) at time t = 0 and returns there at time t = 1/10. Suppose it then repeats that motion nine more times, (so in all it makes 10 circuits, starting at t = 0 and ending at t = 10(1/10) = 1). Let that motion be the curve $\gamma^{**}(t)$.

Consider $\int K(t) - P(t)dt$ over each of the curves $\gamma^{**}(t)$, $\sigma(t) = (0, 0)$, and $\gamma(t) = (0, t^2 - t)$, (all for $0 \le t \le 1$). Over which of those three curves is our integral biggest/smallest? (For $\gamma^{**}(t)$, explain why you can find that integral over $\gamma^*(t)$, and multiply by 10.)

Exercise: Suppose a particle of mass m starts at point A at time t = 0, and then travels in some closed loop, ending up back at A at time t = T. Suppose another particle of the same mass moves twice as fast (as if a video of the first particle was shown at double speed), starts at A at time t = 0 and gets back to A at time t = T/2, and then makes a second trip around the loop again. Argue that $\int P(t)dt$ will be the same for both particles, but $\int K(t)dt$ will be four times larger for the second particle than for the first. How does this argument apply to $\gamma^{**}(t)$ from the previous exercise?

Exercise: Recall in section 2 we had an example in which $P(x, y) = \dfrac{-GMm}{\sqrt{x^2 + y^2}}$, and we showed that $\sigma(t) = (\cos(\sqrt{GM}(t)), \sin(\sqrt{GM}(t))$ is a trajectory whose orbit is a unit circle. For that P(x, y) and that trajectory, find an allowable value for H (as in the above theorem), and the corresponding upper bound on T given by that theorem. Also find the period of that trajectory (the amount of time it takes for the particle to go around the circle once). To what fraction of that circle does the theorem apply?