LOCAL DETERMINISM VERSUS THE KOCHEN-SPECKER THEOREM

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INTRODUCTION: John S. Bell convinced the majority of physicists that local variables do not always exist. In other words, there are certain experiments whose outcomes do not depend solely on factors (also called variables) near enough to the experiment that light from those factors can reach the location of the experiment before it is complete. That means the outcomes of those experiments are either not determined (leaving open the question of how the outcome can appear on a recording device if 'nothing caused it'), or that some of the determining variables involve influences traveling faster than light. Bell's work appears in many places and in many guises. One such, aimed at novices, is [M1]. Despite the fact that Bell's conclusion caused some discomfort amongst physicists, its truth is now routinely accepted. The world is stranger than first thought.

In [KS], Kochen and Specker address the question of determinism. In [CK1] and [CK2], Conway and Kochen use a theorem from [KS] in a somewhat different way to again discuss that question. They claim to show that if humans have a modicum of free will, then some results of experiments in quantum physics are not determined.

There were parts of [CK1] and [CK2] which I was not able to comprehend to the extent that I would be willing to pass judgment on the validity of the full argument. However, the parts of the argument I do feel comfortable with comprise a very nice second proof of the non-existence of local variables. I present it here.

I will spend some time highlighting some subtleties. I hope to show why Conway and Kochen had to do what they did, and that a less involved proof would fail. Having a lesser goal, I can simplify some of their notation.

In their second paper, [CK2], they make four assumptions, SPIN, TWIN, MIN, and a modicum of free will for the experimenter. I will use SPIN and TWIN, but replace MIN by IN, explained later.

To the best of my reading, their use of free will is to assure that the experimental evidence upon which SPIN and TWIN rest is valid, and not the result of a deterministic universe forcing experimenters to never do an experiment revealing that those results are actually false. Those experiments involve the experimenter making some ‘random choices’ of directions. The issue is whether or not the universe allows us to actually make truly random choices.

Remark: When we say the outcome of an experiment is determined, we mean that there are various factors (i.e., variables) that force one and only one outcome. The factors can be complex or subtle. Possibly, some of the factors are unknown to us, leading to the phrase ‘hidden variables’. In an extreme possibility, it is conceivable that some variables are so well hidden that it is impossible to ever find them (despite their existence).

An important distinction is between local and non-local variables. A local variable is a factor close enough to the experiment that light (and hence information) from the factor can reach the experiment prior to its completion. A non-local variable is one so far away that no information from it can reach the experiment in time to influence the outcome. (In the terminology of special relativity, such a variable is said to be space-like separated from the experiment.) Most (but not all) physicists dismiss the possibility of the existence of non-local variables, but the phenomenon of entanglement enticingly dangles the possibility that some influences are superluminal, and can produce an effect without conveying information.

Bells work convinced most physicists that the outcomes of certain experiments are not determined by local variables, meaning that either they are not determined at all, or that the determination involves some superluminal influences. Both possibilities are difficult to imagine, but Bell forced folks to confront them. The present work presents an alternate approach (in some ways easier than Bell’s) to the same conclusion.

Finally, we mention the conondrum associated with an experiment whose outcome is not determined. That means there is a choice of possible outcomes. And yet, the device recording the outcome does produce a single answer. How is it produced? That is a huge mystery.

Here is a possible example. Consider a photon polarized at 45 degrees to the horizontal. If that photon hits a polarizing filter oriented vertically, there is a 50-50 chance it will pass through. Half of such photons get through, and half do not. What is the difference between those that get through and those that do not? It appears quite likely that there is no difference between them! Bell's work shows that whether or not the photon gets through the polarizing filter is not determined by local variables (not even hidden ones). Thus, either some faster than light influence is at work (unlikely according to Einstein), or nothing exists to distinguish the photons that pass through the filter from the ones which do not. The 'choice' of whether the photon gets through or not might very well not be determined by anything! And yet, one choice or the other happens.

We will be discussing an alternate approach to Bells' conclusion, but using electron spin rather than photon polarization.

We need a theorem of Kochen and Specker [KS], giving a proof in Appendix II.

Kochen-Specker Theorem: There is no function f from the set of directions in R^3 to the set {0, 1} with the property that for all 3-tuples of mutually orthogonal directions X, Y, and Z, we have exactly one of f(X), f(Y), and f(Z) is equal to 0.

We will consider a spin-1 particle. (Later, we actually consider a pair of two spin-1 particles, a and b, in an entangled state.) When the component of spin of such a particle is measured, regardless of the direction chosen for the component, the outcome will always be one of ±1 or 0, for the appropriate choice of units. We will drop the word "component", and simply say the spin in any given direction is one of ±1 or 0. (That is an example of the 'discreteness' of the quantum world. In classical physics, it was believed spin components could take any value between the maximum and minimum values.)

While it is not possible to simultaneously measure the spin in two different directions (not even orthogonal directions), it is possible to simultaneously measure the square of spin in any three orthogonal directions. That may seem like a strange pair of facts, and it is, but quantum physics has many strange but true facts.

Definition: By twirl, we will mean the square of spin. (Hence, the twirl in any given direction is either 0 or 1.)

NOTATION: If we measure twirl in the direction w, we will call the result the w-twirl of the particle. If x, y, and z are mutually orthogonal directions, the result of simultaneously measuring twirl in each of them will be called the (x, y, z)-twirl of the particle.
(Herein, (x, y, z) will always represent three mutually orthogonal directions.)

We state three facts predicted by theory and confirmed by experiment. Either of the first two closely related facts is often called the 101-law. Consistent with Conway and Kochen, we will call our facts SPIN, SPIN′, and SPIN″.

SPIN: If twirl is simultaneously measured in three orthogonal directions, exactly two of those twirls will be 1 and the third will be 0.

SPIN′: If three twirl measurements are made sequentially, one after the other, in three mutually orthogonal directions, with no other measurements made in between, exactly two of those twirls will be 1 and the third will be 0.

SPIN″: If w and u are non-orthogonal directions, and if you first measure the w-twirl, then measure the u-twirl, and finally measure the w-twirl again, the first and last readings for the w-twirl might not be equal. Thus, making a twirl measurement (such as the u-twirl) changes the particle, so that previously known twirl values are no longer valid. (This is a case of the general rule of thumb that an observation might change that which is observed.)

We wish to consider the question of whether twirl measurements have determined outcomes, or if the there is some freedom of choice in the outcome. (If the latter, we avoid the mind-curdling question of how the choice is made.)

ASSUMPTION (which may well be false): The result of any twirl measurement is determined.

Assuming there is no choice in the outcome, we can define functions as follows.

NOTATION: Let $S(w) \in \{0,1\}$ be the result of measuring the w-twirl of the particle. Let $T(x, y, z) \in \{(1, 1, 0), (1, 0, 1), (0, 1, 1)\}$ be the result of measuring the (x, y, z)-twirl of the particle. (S and T stand for 'single' and 'triple'.)

Following the lead of [CK1], we start with an easy argument following immediately from SPIN and the Kochen-Specker theorem. We begin by pointing out that twirl is the number produced by the twirl measurement, just as temperature is the number on the thermometer. To say that twirl exists prior to its measurement is equivalent to saying that the outcome of the measurement is determined, and the measurement itself is not one of the determining factors.

THEOREM A: If SPIN is true, then twirls do not exist prior to their measurement.

Proof: Suppose the answer to a twirl measurement existed prior to the measurement. Then the answer is determined and so the functions S and T exist, and furthermore we would have $T(x, y, z) = (S(x), S(y), S(z))$ for any mutually orthogonal x, y, and z. However, SPIN then shows that triple consist of two ones and a zero. That shows S contradicts the Kochen-Specker theorem.

Since SPIN is almost surely true, Theorem A shows that if twirl measurements are determined, then one of the determining factors is the act of taking the measurement. That is not too surprising. Taking a measurement involves setting up an apparatus whose presence can affect the particle. The choice of which direction w is being measured affects the apparatus, and hence the outcome. Thus, even if twirl measurements are not entirely determined, it is still believable that the measurement itself might have some influence over the answer. After all, taking a temperature affects the temperature, since the thermometer's own temperature modifies the temperature being measured.

Our hope is to again use the Kochen-Specker theorem to prove the stronger result that S cannot exist at all. (We will not quite achieve that goal, but rather will show that if it does exist, there must be non-local factors at play.) It will be instructive to begin with three flawed arguments. Examining why they fail will show us why we must make use of entangled particles.

FLAWED ARGUMENT 1: SPIN′ shows that if one successively measures the x-twirl, then the y-twirl, then the z-twirl of our particle, exactly one of those answers is 0. At first glance, that would appear to mean the alleged function S(w) violates the Kochen-Specker theorem. However, SPIN′′ shows each successive measurement changes the particle, and so might change the function S. If S actually exists, then the triple (S(x), S(y), S(z)) exists as a mathematical entity, but that does not mean we can find it via successive measurements. The successive measurements will give (S_1(x), S_2(y), S_3(z)) for three evolving versions of S. Even though experiments show (S_1(x), S_2(y), S_3(z)) ∈ {(1, 1, 0), (1, 0, 1), (0, 1, 1)} (i.e., SPIN′), we do not violate the Kochen-Specker theorem, since we have three functions, not one.

FLAWED ARGUMENT 2: Assuming T exists, SPIN says T(x, y, z) ∈ {(1, 1, 0), (1, 0, 1), (0, 1, 1)}. However, we cannot hope to violate the Kochen-Specker theorem by using that fact directly, since that theorem discusses a function whose range is {0, 1}, not {(1, 1, 0), (1, 0, 1), (0, 1, 1)}.

FLAWED ARGUMENT 3: Let us use T to find S. Thus, to find S(w), pick a mutually orthogonal triple (x, y, z) with w ∈ {x, y, z}, and then take S(w) to be the w-twirl given by T(x, y, z). Then by definition, we have (S(x), S(y), S(z)) = T(x, y, z), and so SPIN shows S contradicts the Kochen-Specker theorem. The problem here is that S might not be well-defined. So far, we have no way of knowing that T(x, y, z) and T(x, y′, z′) both give the same value to the x-twirl. Taking a triple measurement requires certain equipment oriented in a certain way, interacting with the particle. We already know the measurement influences the answer. Different orientations of the equipment might give different x-twirls.

SPIN shows T(x, y, z) ∈ {(1, 1, 0), (1, 0, 1), (0, 1, 1)}, while SPIN′ shows (S_1(x), S_2(y), S_3(z)) ∈ {(1, 1, 0), (1, 0, 1), (0, 1, 1)}. Neither of those contradicts the Kochen-Specker theorem. For that, we would need (S(x), S(y), S(z)) ∈ {(1, 1, 0), (1, 0, 1), (0, 1, 1)}, and none of the experimental evidence mentioned so far allows us to say that. Soon, we will get around that problem by showing that *under appropriate assumptions*, (S(x), S(y), S(z)) = T(x, y, z) ∈ {(1, 1, 0), (1, 0, 1), (0, 1, 1)}. We will then consider the reliability of the needed assumptions.

As seen in the third false argument, we cannot blithely assume (S(x), S(y), S(z)) = T(x, y, z), since the equipment used and its orientation matters. (We did use that equality in the proof of Theorem A, but that was based on the incorrect assumption that the act of measuring did not affect the answer.)

In order to proceed, we introduce a pair of spin-1 particles, called particle a and particle b, along with their respective experimenters A and B.

CONVENTIONS: When we speak of T(x, y, z), it refers to particle a, while S(w) refers to particle b.

We previously spoke of S and T as if they applied to the same particle. Now they apply to different particles. Can we still speak of S(w) and T(x, y, z), or must we discuss S(x, y, z, w) and T(x, y, z, w)? Does A's act of measurement influence B's answer and vice-versa? Let us introduce a new function (still assuming twirl measurements are determined).

NOTATION: C(x, y, x, w) is the function from the set of all 4-tuples (x, y, z, w) such that x, y, and z are mutually orthogonal, to the set {(1, 1, 0, 0), (1, 1, 0, 1), (1, 0, 1, 0), (1, 0, 1, 1), (0, 1, 1, 0), (0, 1, 1, 1)}, giving the combined outcome of A doing a (x, y, z)-twirl measurement on particle a and B doing a w-twirl measurement on particle b.

CRUCIAL QUESTION: Does $C(x, y, z, w) = (T(x, y, z), S(w))$? (Here we use the obvious interpretation of that last expression. C stands for 'combined'.)

The reader will have no difficulty seeing that if the result of A's measurement is completely independent of B's action (or inaction, or mere presence), and vice-versa, then we will have $C(x, y, z, w) = (T(x, y, z), S(w))$. On the other hand, if there is some dependence between A and B, then we cannot even be sure that $C(x, y, z, w) = (T'(x, y, z), S'(w))$ for any functions S', and T'. Perhaps the value of the first three components of the answer $C(x, y, z, w)$ depends on w as well as on (x, y, z), and perhaps the value of the fourth component of $C(x, y, z, w)$ depends not only on w, but on (x, y, z) as well. If that is the case, then our previous functions S and T do not exist. After all, S and T were discussed under the tacit assumption that there was a single particle and a single experimenter. Conceivably, they do exist in that situation, but cease to exist upon introduction of a second particle and experimenter, and need to be replaced by C (so that we still have determinism, despite the loss of S and T).

To avoid such pedantry, we now present a situation in which most physicists would accept that A's results are totally independent of what, if anything, B is doing, and vice-versa.

Suppose A and B are far apart, but agree to take their measurements at the same time. Then those two measurements are space-like separated events, and so special relativity tells us there are frames of reference in which A acted before B (so apparently nothing B does can affect A's result), and other frames of reference in which B acted before A (so apparently nothing A does can affect B's result). We also assume there is no prior collusion between A and B concerning what measurement, if any, they will make, but rather, if they do choose to make a measurement, the directions involved will be chosen randomly. (Prior collusion would constitute a local variable they carried with them.)

We make a new assumption based on the preceding comments.

IN: The IN assumption says that in the situation just described, S, T, and C exist, and $C(x, y, z, w) = (T(x, y, z), S(w))$. ('IN' stands for 'independence'.)

Notice that assuming IN implies S, T and C exist, which in turn implies twirl measurements are determined. As for the converse, if twirl measurements are locally determined, then IN holds. However, if twirl measurements are non-locally determined, IN may or may not hold, depending on the exact nature of the non-local variables. Surely it will not hold if one of those variables is what the far distant other experimenter is doing.

We now come to the most important part of the argument. We will add the assumption that our particles a and b are entangled in a way we now describe, called TWIN. (The reader not familiar with entanglement can find an introduction in [M2]. That work deals with polarization of photons instead of spins of particles, but the principles of entanglement are the same.)

TWIN: One can create a pair of entangled spin-1 particles a and b, such that if an (x, y, z)-twirl of particle a is measured by experimenter A, and a w-twirl of particle b is measured by experimenter B, then the following is true. If $w \in \{x, y, z\}$, then the twirl assigned to w by the triple measurement of particle a equals the twirl assigned to w by the singleton measurement of particle b. Furthermore, this is true even if the actions of A and B are space-like separated, and do not involve prior collusion between A and B. (Caution: There is a subtle, but major problem concerning that last statement. In Appendix I we will discuss and rectify it.)

LEMMA: Suppose TWIN and IN are true. Then $T(x, y, z) = (S(x), S(y), S(z))$.

Proof: Assuming IN shows S, T, and C exist. Suppose $C(x, y, z, w) = (\alpha, \beta, \gamma, \delta)$. By IN, we have $T(x, y, z) = (\alpha, \beta, \gamma)$ and $S(w) = \delta$. Consider the case that $w = x$. Then TWIN tells us $S(x) = \alpha$. Similarly, letting $w = y$, TWIN tells us $S(y) = \beta$, and letting $w = z$, we get $S(z) = \gamma$. Thus $T(x, y, z) = (S(x), S(y), S(z))$.

THEOREM B: SPIN, TWIN, and IN cannot all be true.

Proof: Suppose SPIN, TWIN, and IN all hold. Then by the lemma and SPIN, $(S(x), S(y), S(z)) = T(x, y, z) \in \{(1, 1, 0), (1, 0, 1), (0, 1, 1)\}$. That contradicts the Kochen-Specker theorem.

Theorem B shows that at most two of SPIN, TWIN, and IN can be true. Which is most likely to be false? SPIN is based on a lot of experimental evidence. It is also predicted by theory, but experimental verification always outranks theoretical prediction. As mentioned in the introduction, it is possible that SPIN is actually false, but that we live in a deterministic universe in which we are constrained to never do an experiment revealing its falsehood. That determinism might also lead us to create false theories based on deceitful experiments.

TWIN also rests upon theoretical prediction and experimental verification. However, we have been using TWIN in a situation in which A and B are space-like separated and have no prior collusion. That makes our version of TWIN extremely hard to verify experimentally. In Appendix I, we explain that statement, and show how to circumvent the problem. We will see that a slight change of context makes our version of TWIN experimentally verifiable.

The weakest assumption is IN. First of all, it incorporates two sub-assumptions, the first being that twirl measurements are determined, and the second being independence between A's results and B's results. That second sub-assumption is based on the theory of special relativity, and is not amenable to experimentation.

In view of the above, it is probably wisest to restate Theorem B as follows.

PROPOSITION: Suppose SPIN and TWIN are true. Then IN is false. Hence, either the outcomes of twirl measurements are not always determined (i.e., C does not exist), or the measurements made by A and B are not independent of one another, even under the assumption of space-like separation without prior collusion.

Let us assume SPIN and TWIN, as most physicists would probably do. Then IN is false. Thus, at least one of its sub-assumptions is false. Which? I suspect that most modern physicists would say that the outcomes of twirl measurements are not determined, but the measurements made by A and B are independent of each other.

Personally, I am willing to consider the possibility that both sub-assumptions of IN are false. I feel that we should not be over hasty in accepting the belief that the results of A and B are independent. That belief is based on the fact that there are frames of reference in which A acted before B, and other frames of reference in which B acted before A. Thus, it is argued, nothing A does can force B's result, and vice-versa. However, the word 'and' in that last sentence hides a change of frame of reference.

The observer who sees B acting first says that nothing A does can change B's answer. However, that observer could argue that B's action does influence A's answer. The other observer, who sees A acting first, can say that A's action influenced B's answer. Different observers have different views of time and place, and so why can they not also have different views of cause and effect relationships?

EXAMPLE: Suppose measurements C (cause) and E (effect) can both take values either 0 or 1. Suppose experiments show we either get (C, E) = (0, 0), or (C, E) = (1, 1). We could argue that C causes E. However, suppose an observer sees E as occurring first. To that person, E causes C. Both views explain the data, and neither puts an effect before a cause.

Another reason I hesitate about accepting independence between A and B is the fact that we know so little about 'cause and effect'. (I have never seen a definition.) Must that relationship depend purely on a transfer of information from the cause to the effect, or can non-informational influences, some possibly being superluminal, be involved?

The proof of our proposition shares an important commonality with Bell's proof. Both utilize the tension between entanglement and special relativity. The correlated behavior between two entangled particles argues in favor of dependence. Special relativity argues in favor of independence. In both proofs, the resolution of that tension is to conclude that if determinism exists, it involves non-local variables. ([M2] explores why that tension does not lead to out and out conflict.)

In [KS], the authors apparently use the Kochen-Specker theorem to argue against the existence of local determining variables. I have not read that paper. It appears to be somewhat more abstract in its approach than what we have done here.

David Bohm developed a theory of quantum physics using non-local variables. Although I am not familiar with it, I have read that it is not compatible with special relativity.

In [CK2, Appendix], the authors claim that their arguments show no non-local theory such as Bohm's can ever be relativistic. However, I personally was not able to follow their reasoning.

APPENDIX I

TWIN is predicted by standard theory. However, if we want our arguments to be as free from theory as possible, and just rest upon experimentally proven facts, we have a problem with the statement that the TWIN property holds even if the measurements of A and B are space-like separated and do not involve prior collusion between A and B. Here is why.

An efficient way to test TWIN would be to have A measure some (x, y, z)-twirl, and have B measure a w-twirl where w is one of x, y, or z. However, achieving that last would require either some current communication or prior collusion between A and B. We have excluded those possibilities, and are assuming A and B make random choices. Given an infinitude of directions to choose from, the odds that w will be one of x, y, or z is vanishingly small. TWIN only talks about cases where $w \in \{x, y, z\}$, and since that would almost never occur, we would not collect enough data to reliably verify TWIN. (Determining whether w was in $\{x, y, z\}$ is part of the post experiment analysis, involving later communication between A and B.)

Essentially, in this appendix, we are considering the possibility that experiments which allow communication or prior collusion between A and B do verify TWIN, but that when communication and prior collusion are eliminated, TWIN experiments *might* fail.

Can we devise an experiment designed to eliminate that possibility, showing that TWIN always holds)? Yes! To do so, we need a stronger version of the Kochen-Specker theorem.

Finite Kochen-Specker theorem: There is a set V comprising 57 directions, such that there is no function f from the set of directions in V to the set {0, 1} with the property that for all 3-tuples of directions (x, y, z) with x, y, and z mutually orthogonal and coming from V, we have exactly one of f(x), f(y), and f(z) is equal to 0.

We now specify that in all our previous discussions, we insist that any direction mentioned comes from V. When A and B make their choices, it will still often happen that $w \notin \{x, y, z\}$ (which does not help us verify TWIN), but since V is finite, with enough trials in our experiment, we will eventually find many cases in which w is in {x, y, z}. Those cases will allow us to verify TWIN experimentally.

I doubt that the experiment just described, only using directions from V, has actually been done. I have no doubt that were it done, TWIN would be verified. After all, it is predicted by a highly reliable theory.

Remark: Of course, when we restrict directions to those in V, we are no longer working with the original TWIN, but with TWIN restricted to V. However, having the restricted version of TWIN hold for V is good enough to reach our desired goal that if twirl measurements are determined, then there are non-local variables involved.

I found it difficult to discern if the authors of [CK2] concerned themselves about the issue raised in this appendix. They do not mention it overtly. They do mention a finite version of the Kochen-Specker theorem, but using a stronger hypothesis then ours. It involves a set W of 33 directions, and they later briefly mention augmenting W to a larger set for which our Finite Kochen-Specker theorem would hold. They also mention a finite set of possible (x, y, z, w).

APPENDIX II

We now prove the Finite Kochen-Specker theorem.

Notation: By [x, y, z], we will mean the direction indicated by the vector (x, y, z). We specify that [x, y, z] = [-x, -y, -z], despite the fact that (x, y, z) and (-x, -y, -x) are sometimes said to point in opposite directions.

Notation: Let K[x, y, z] =
{[±x, ±y, ±z], [±x, ±z, ±y], [±y, ±x, ±z], [±y, ±z, ±x], [±z, ±x, ±y], [±z, ±y, ±x]}.

Notation: Let V =
K[1, 0, 0] ∪ K[1, 1, 0] ∪ K[1, $\sqrt{2}$, 0] ∪ K[1, $\sqrt{2}$/2, $\sqrt{2}$/2] ∪ K[1, $\sqrt{2}$/2, 3$\sqrt{2}$/2].

Let us find the size of V. It is not hard to see that the above union is disjoint. Since a vector and its negative give the same direction, we see that K[1, 0, 0] contains 3 directions. Now there are 6 permutations of (1, $\sqrt{2}$, 0), all distinguishable from each other. For each of them, there are two choices for ±1 and two choices for $\pm\sqrt{2}$ (but only one choice of ±0) giving 24 vectors. However, a vector and its negative give the same direction, and so K[1, $\sqrt{2}$, 0] contains 12 directions. The 6 permutations of (1, 1, 0) produce only 3 distinct vectors. Each of the 2 nonzero entries has 2 choices of sign. Thus we get $(3)(2^2)/2 = 6$ directions. The 6 permutations of (1, $\sqrt{2}$/2, $\sqrt{2}$/2) also only produce 3 distinguishable vectors, but in each one, there are 2 choices for the sign of each of the 3 coordinates. We have $(3)(2^3)/2 = 12$ directions. Finally, K[1, $\sqrt{2}$/2, 3$\sqrt{2}$/2] contains 24 directions. Therefore, the size of V is 57. (I have not considered the question of the smallest possible V for which the Finite Kochen-Specker theorem holds. The V used here is easy to work with.)

Notation: If X, Y, and Z are directions, we will write <X, Y, Z> to mean that X, Y, and Z are mutually orthogonal.

Finite Kochen-Specker theorem: There is no function f from the set V to {0, 1} such that if X, Y, and Z are in V and if <X, Y, Z> then exactly one of f(X), f(Y), and f(Z) is 0. (Obviously the Kochen-Specker theorem follows from the Finite Kochen-Specker theorem via restriction to V.)

Proof: The reader can easily see that the following 25 directions are all contained in V.

D_1 [1, 0, 0] D_2 [0, 1, 0] D_3 [0, 0, 1] D_4 [0, 1, 1] D_5 [0, 1, -1]

D_6 [1, 0, 1] D_7 [1, 0, -1] D_8 $[1, \sqrt{2}, 0]$ D_9 $[1, -\sqrt{2}, 0]$ D_{10} $[\sqrt{2}, 1, 0]$

D_{11} $[\sqrt{2}, -1, 0]$ D_{12} $[1, \sqrt{2}/2, \sqrt{2}/2]$ D_{13} $[1, -\sqrt{2}/2, -\sqrt{2}/2]$

D_{14} $[1, \sqrt{2}/2, -\sqrt{2}/2]$ D_{15} $[1, -\sqrt{2}/2, \sqrt{2}/2]$ D_{16} $[\sqrt{2}/2, 1, \sqrt{2}/2]$

D_{17} $[\sqrt{2}/2, -1, -\sqrt{2}/2]$ D_{18} $[\sqrt{2}/2, -1, \sqrt{2}/2]$ D_{19} $[\sqrt{2}/2, 1, -\sqrt{2}/2]$

D_{20} $[1, -\sqrt{2}/2, -3\sqrt{2}/2]$ D_{21} $[1, \sqrt{2}/2, -3\sqrt{2}/2]$ D_{22} $[\sqrt{2}/2, 1, 3\sqrt{2}/2]$

D_{23} $[\sqrt{2}/2, 1, -3\sqrt{2}/2]$ D_{24} $[\sqrt{2}/2, -1, -3\sqrt{2}/2]$ D_{25} $[\sqrt{2}/2, -1, 3\sqrt{2}/2]$

The proof is by contradiction, so suppose f exists. Let Q be the permutation of directions that sends [x, y, z] to [z, x, y]. The definition of K[x, y, z] shows that V is closed under Q. Also, Q obviously preserves inner products, and so preserves orthogonality. Therefore $f \circ Q$ also satisfies the hypothesis of the theorem, as does $f \circ Q^2$.

Since $\langle D_1, D_2, D_3 \rangle$, f sends one of those three directions to 0. Since $Q(D_1) = D_2$ and $Q^2(D_1) = D_3$, the previous paragraph shows we may assume $f(D_1) = 0$.

Since $<D_1, D_4, D_5>$ the hypothesis shows we have $f(D_4) = 1 = f(D_5)$. Since $<D_4, D_{14}, D_{15}>$ and $f(D_4) = 1$ we must have that exactly one of D_{14} or D_{15} is sent to 0. Similarly, since $<D_5, D_{12}, D_{13}>$ and $f(D_5) = 1$, we see that exactly one of D_{12} or D_{13} is sent to 0. Combining those possibilities, we see there are four possible cases, and they are distinguished by saying ($D_{15}\to 0$, $D_{12}\to 0$), or ($D_{14}\to 0$, $D_{12}\to 0$), or ($D_{15}\to 0$, $D_{13}\to 0$), or ($D_{14}\to 0$, $D_{13}\to 0$). We will now use certain symmetries to eliminate all four cases.

Suppose P is a permutation of the vectors in V that preserves orthogonality, and fixes D_1. Writing PX for P(X), we thus have $PD_1 = D_1$, and $<X, Y, Z>$ implies $<PX, PY, PZ>$.

Claim: It is impossible to have both $f(PD_{15}) = 0$ and $f(PD_{12}) = 0$.

Suppose $f(PD_{15}) = 0 = f(PD_{12})$. Since $<D_1, D_2, D_3>$ and $PD_1 = D_1$, we have $<D_1, PD_2, PD_3>$. Since $f(D_1) = 0$, we must have $f(PD_3) = 1$. Since $<D_{15}, D_8, D_{20}>$, we have $<PD_{15}, PD_8, PD_{20}>$, and so since $f(PD_{15}) = 0$, we must have $f(PD_8) = 1$. Since $< D_3, D_8, D_{11}>$ we have $<PD_3, PD_8, PD_{11}>$, and so since we know $f(PD_3) = 1 = f(PD_8)$, we have $f(PD_{11}) = 0$. Since $<D_{11}, D_{19}, D_{22}>$ and $<D_{11}, D_{16}, D_{23}>$, having $f(PD_{11}) = 0$ forces $f(PD_{19}) = 1 = f(PD_{16})$.

Since $< D_{12}, D_9, D_{21}> = 0$ and $f(PD_{12}) = 0$, we must have $f(PD_9) = 1$. Since $<D_3, D_9, D_{10}>$ with $f(PD_3) = 1 = f(PD_9)$, we must have $f(PD_{10}) = 0$. Since $<D_{10}, D_{18}, D_{24}>$ and $<D_{10}, D_{17}, D_{25}>$, we must have $f(PD_{18}) = 1 = f(PD_{17})$.

Now $< D_7, D_{16}, D_{18}>$ and $f(PD_{16}) = 1 = f(PD_{18})$, so that $f(PD_7) = 0$. Similarly, $<D_6, D_{19}, D_{17}>$ and $f(PD_{19}) = 1 = f(PD_{17})$ forces $f(PD_6) = 0$.

As $< D_7, D_6, D_2>$, we have $<PD_7, PD_6, PD_2>$, contradicting that f sends both PD_7 and PD_6 to 0.

We now specify four choices for the permutation P. The definition of K[x, y, z] makes it clear that each of our four choices of P is a permutation of V. Each fixes D_1, and each preserves inner products and so preserves orthogonality.

First consider the case that $P = P_1$ is the identity map. That tells us it is impossible for f to send D_{15} and D_{12} both to 0. (That eliminates the earlier case ($D_{15}\to 0$, $D_{12}\to 0$).) Next, let

$P = P_2$ be the map sending [x, y, z] to [x, z, y]. That shows us it is impossible for f to send $D_{14} = P_2D_{15}$ to 0 and $D_{12} = P_2D_{12}$ to 0. (That eliminates the earlier case ($D_{14} \to 0$, $D_{12} \to 0$).)
Now let $P = P_3$ send [x, y, z] to [x, -z, -y]. We conclude that it is impossible for f to send $D_{15} = P_3D_{15}$ to 0 and $D_{13} = P_3D_{12}$ to 0. (That eliminates the case ($D_{15} \to 0$, $D_{13} \to 0$).)
Finally, let $P = P_4$ send [x, y, z] to [x, -y, -z]. We now learn that it is impossible for f to send $D_{14} = P_4D_{15}$ to 0 and $D_{13} = P_4D_{12}$ to 0. (That eliminates the case ($D_{14} \to 0$, $D_{13} \to 0$).)

Remark: The Finite Kochen-Specker theorem is not quite proven in [CK2], although clues are given as to how to get to it from their proof of the Kochen-Specker theorem. Since we need the finite form of the theorem (to rectify the problem mentioned in Appendix I), we followed those clues.

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