The Growth of Money

1. Interest

2. Accumulation and amount functions

3. Simple Interest/Linear Accumulation Functions

4. Discount functions/The time value of money

5. Simple discount
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What is interest?

• $K$...the principal, i.e., the amount of money that the **borrower** borrows/lender lends at time $t = 0$

• $S$...the amount of money that changes hands at a later time - say, $T$

• The interest is defined as

$$S - K \geq 0$$

*Reading assignment:* Section 1.2 in the textbook (on the rationale behind the existence of interest)
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The amount function

- Let us temporarily fix the principal $K$
- $A_K(t)$ ... the amount function for principal $K$, i.e., the balance at time $t \geq 0$ (time is always measured in some agreed upon units; think “years” for now)
- In words: $K$ invested at time $t = 0$ “grows” to $A_K(t)$ at time $t \geq 0$
- Note: $A_K(0) = K$
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The accumulation function

• $a(t)$ . . . the accumulation function, i.e., the amount function if the principal $K$ is one dollar

• Formally: If the principal is one dollar, we write

$$a(t) = A_1(t)$$

• Note:

$$a(0) = 1$$
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The relationship between the amount and the accumulation functions

- We expect to have that
  \[ A_K(t) = Ka(t) \]

- This is common, but is NOT always the case (the investment scheme may include a tiered growth structure).
- However, since the above equality holds in most cases, we will assume that it is true unless it is explicitly noted otherwise.
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The increase/decrease of the amount and the accumulation functions

• It is natural to assume that both $a$ and $A_K$ increase in the time variable.
  • Such increase may be, for example:
    ◦ continuous and linear;
    ◦ discrete (end of the year, e.g.);
    ◦ continuous and exponential
  • However, there are investment schemes in which it is possible to lose money over time (e.g., if one invests in a fund that trades in the market or in a restaurant that takes time to pay off)
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The effective interest rate

Let \( t_2 > t_1 \geq 0 \)

- \( A_K(t_2) - A_K(t_1) \) ... the amount of interest earned between time \( t_1 \) and time \( t_2 \)
- \( i_{[t_1,t_2]} \) ... the effective interest rate for the interval \([t_1, t_2] \), i.e.,
  \[
i_{[t_1,t_2]} = \frac{a(t_2) - a(t_1)}{a(t_1)}\]

- **IF** \( A_K(t) = Ka(t) \), then we also have
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- The interval \([n-1, n]\) is called the \( n^{th} \) time period (for \( n \) a positive integer)
- **Notation:**
  \[
i_n = i_{[n-1,n]} = \frac{a(n) - a(n-1)}{a(n-1)}\]

- Hence,
  \[
a(n) = a(n-1)(1 + i_n)\]

and \( i_1 = a(1) - 1 \)
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In this case, \( a \) is assumed linear and, thus, must be of the form

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a(t) = 1 + st
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for a certain constant \( s \)

- \( s \) ... the simple interest rate
- Note: \( s = i_1 \)
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- Let us look at an example ...
Linear $a(t)$

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On \( i_n \)

- In the simple interest case:

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i_n = \frac{s}{1 + s(n - 1)}
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- So, \( i_n \) is decreasing in \( n \) (see Example 1.4.2 in the textbook for an illustration of this fact)

- Moreover,

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i_n \to 0, \text{ as } n \to \infty
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- So, simple interest is not convenient for long duration loans.
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Methods for measuring the time/length of the loan in years

- **Exact simple interest aka ”actual/actual”**
  
  The loan term $D$ expressed in days and divided by 365

- **Ordinary simple interest aka ”30/360”**
  
  The loan term $D$ expressed in days assuming that each month has 30 days and then divided by 360

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• In words, $v(t)$ is the amount of money that one should invest at time 0 in order to have $1 at time $t$
• For example, in the simple interest case, we have that

$$v(t) = \frac{1}{1 + st}$$

• Question: What if one wishes to invest a certain amount not at time 0 but at a later time $t_1 > 0$ - with the goal of earning $S$ at a still later time $t_2$?
• Let us draw the time line ....
• One needs to invest (at time $t_1$)

$$Sv(t_2)a(t_1) = S\frac{a(t_1)}{a(t_2)} = S\frac{v(t_2)}{v(t_1)}$$

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$$v(t) = \frac{1}{a(t)}$$

• In words, $v(t)$ is the amount of money that one should **invest** at time 0 in order to have $1 at time $t$

• For example, in the simple interest case, we have that

$$v(t) = \frac{1}{1 + st}$$

• **Question:** What if one wishes to invest a certain amount not at time 0 but at a later time $t_1 > 0$ - with the goal of earning $S$ at a still later time $t_2$?

• Let us draw the time line ....

• One needs to invest (at time $t_1$)

$$Sv(t_2)a(t_1) = S\frac{a(t_1)}{a(t_2)} = S\frac{v(t_2)}{v(t_1)}$$

• **Assignment:** Examples 1.7.2, 1.7.3 in the textbook
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• \( PV_{a(t)}(\$L \text{ at } t_0) \) ... present value with respect to \( a(t) \) of \( \$L \) to be received at time \( t_0 \), i.e.,

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if the growth is proportional to the invested amount

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The Growth of Money

1. Interest
2. Accumulation and amount functions
3. Simple Interest/Linear Accumulation Functions
4. Discount functions/The time value of money
5. Simple discount
Discount rate

- $D \ldots$ the **discount** per unit time period, per dollar that the borrower and the lender agree upon at time 0, i.e.,

  If an investor (lender) lends $1$ for one basic period at a discount rate $D$ - this means that in order to obtain $1$ at time 0, the borrower must **pay immediately** $D$ to the lender.

- Note that the “net-effect” for the borrower is that they get to use 

  $$$(1 - D)$$

  at time zero

- The initial fee is proportional to the amount of money borrowed, i.e., if one wants to borrow $K$, one needs to pay $DK$ to the lender

- The value $DK$ is called the **amount of discount**
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or, equivalently

$$v(t) = 1 - tD$$

- Note that the discount function $v(t)$ is linear in this case
- **Caveat:** This situation is not the same as the one when the accumulation function is linear.
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$A_K(t)$ and $a(t)$

- More vocabulary:

\[ A_K(t) = \frac{K}{1 - dt} \]

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- Let us draw their graphs .....  
- Note that it only makes sense to talk about loan terms that are shorter than $1/d$
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