Equilibrium Short-Rate Bond Price Models

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Delta-Gamma Approximations for Bonds

- Recall the equilibrium equation (see (24.18) in the textbook):

$$\frac{1}{2}\sigma(r)^2\frac{\partial^2 P}{\partial r^2} + [\alpha(r) - \sigma(r)\phi(r,t)]\frac{\partial P}{\partial r} + \frac{\partial P}{\partial t} - rP = 0$$

where

1. r denotes the short-term interest rate, which follows the Ito process

$$dr = a(r)dt + \sigma(r)dZ;$$

2. $\phi(r,t)$ is the Sharpe ratio corresponding to the source of uncertainty Z, i.e.,

$$\phi(r,t) = \frac{\alpha(r,t,T) - r}{q(r,t,T)}$$

with the coefficients $P \cdot \alpha$ and $P \cdot q$ are the drift and the volatility (respectively) of the Ito process P which represents the bond-price for the interest-rate r

- Due to Ito's Lemma and the equilibrium equation we have the following heuristic expression (see (24.22) in the textbook)

$$\mathbb{E}^*[\frac{dP}{dt}] = \frac{1}{2}\sigma(r)^2\frac{\partial^2 P}{\partial r^2} + [\alpha(r) - \sigma(r)\phi(r,t)]\frac{\partial P}{\partial r} + \frac{\partial P}{\partial t} = rP,$$

i.e., under the risk-neutral measure the bonds earn the risk-free rate

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Delta-Gamma-Theta Approximation

- It is natural to interpret the equation

$$\frac{1}{2}\sigma(r)^2\frac{\partial^2 P}{\partial r^2} + [\alpha(r) - \sigma(r)\phi(r,t)]\frac{\partial P}{\partial r} + \frac{\partial P}{\partial t} = rP$$

  as the Delta-Gamma-Theta approximation

Equilibrium Short-Rate Bond Price Models

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- We discuss three bond pricing models based on the equilibrium equation, in which all bond prices are driven by the short-term interest rate r:
- The Rendleman-Bartter model
- The Vasiček model
- The Cox-Ingersoll-Ross model
- They differ in their specification of the coefficients of the SDE that the short-term interest rate is required to satisfy

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An Overly Simple Model

- The simplest models of the short-term interest rate are those in which the interest rate is modeled as an arithmetic or a geometric Brownian motion
- For example:

$$dr = a\,dt + \sigma\,dZ$$

- In this specification, the short-rate is normally distributed with mean $r_0 + at$ and variance $\sigma^2 t$
- There are several shortcomings of this model:
- The r can become negative
- The drift in the SDE for r is constant - meaning that for $a > 0$, r is expected to drift to ∞ over a long time
- The volatility of the short-rate is the same whether the rate is high or low - this clashes with the practical observation that higher rates tend to be more volatile

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The Rendleman-Bartter model

- The Rendleman-Bartter model assumes that the short-rate follows a geometric Brownian motion, i.e., it satisfies the following SDE:

$$dr = ar\,dt + \sigma r\,dZ$$

- An objection to this model is that interest rates can be arbitrarily high. In practice, we would expect rates to exhibit mean reversion

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The Vasiček model

- The Vasicek model incorporates mean reversion, i.e., it is defined as an Ornstein-Uhlenbeck process

$$dr = a(b - r)\,dt + \sigma\,dZ$$

This is an Ornstein-Uhlenbeck process.

- The $a(b - r)\,dt$ term induces mean reversion
- However, it is possible for interest rates to become negative and the variability of interest rates is independent of the level of rates

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The Vasiček model: The Expression for the Bond-price

- Under the assumption that $a \neq 0$, the equilibrium equation reads as

$$\frac{1}{2}\sigma^2 \frac{\partial^2 P}{\partial r^2} + [a(b-r) - \sigma\phi]\frac{\partial P}{\partial r} + \frac{\partial P}{\partial t} - rP = 0$$

with the boundary condition $P(T, T, r) = 1$

- The formula for the bond-price is then

$$P[t, T, r(t)] = A(t, T)e^{-B(t,T)r(t)}$$

where

$$A(t, T) = exp\{\bar{r}(B(t, T) - (T - t)) - \frac{B^2\sigma^2}{4a}\}$$

$$B(t, T) = \frac{1}{a}(1 - e^{-a(T-t)})$$

$$\bar{r} = b + \frac{\sigma\phi}{a} - \frac{\sigma^2}{2a^2}$$

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The Cox-Ingersoll-Ross model

- The Cox-Ingersoll-Ross model:

$$dr = a(b - r)\,dt + \sigma\sqrt{r}\,dZ$$

- It is impossible for interest rates to be negative
- The $a(b - r)\,dt$ term again induces mean reversion
- As the short-rate rises, the volatility of the short-rate also rises
- The model rectifies all the objections to the earlier models

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The CIR model: The Expression for the Bond-price

- Under this model, the risk premium takes the form $\phi = \frac{\bar{\phi}\sqrt{r}}{\sigma}$ and with this notation, the equilibrium equation reads as

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$$B(t, T) = \frac{2(e^{\gamma(T-t)} - 1)}{(a + \bar{\phi} + \gamma)(e^{\gamma(T-t)} - 1) + 2\gamma}$$

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