Bond Options, Caps and the Black Model

Black formula

- Recall the **Black formula** for pricing options on futures:

$$C(F, K, \sigma, r, T, r) = Fe^{-rT}N(d_1) - Ke^{-rT}N(d_2)$$

where

$$d_1 = \frac{1}{\sigma\sqrt{T}}\left[\ln(\frac{F}{K}) + \frac{1}{2}\sigma^2 T\right]$$
$$d_2 = d_1 - \sigma\sqrt{T}$$

Options on Bonds: The set-up

- Consider a call option on a zero-coupon bond paying \$1 at time $T+s$. The maturity of the option is T and the strike is K.
- The payoff of the above option is

$$(P(T,T+s)-K)^+$$

where $P(T,T+s)$ denotes the price of the bond (maturing at $T+s$) at time T

- **Questions:**
How do we apply the Black-Scholes setting to the above option? What are the correct assumptions that are analogues of the lognormallity we imposed on the prices of the underlying asset in the Black-Scholes pricing model?

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Exchange Options: The definition and set-up

- It turns out that the convenient tool for solving the above problem is to recast the set-up in terms of a particular family of **exotic options**, namely, exchange options.
- An exchange option pays off only if the underlying asset outperforms some other asset (**benchmark**). Hence, these options are also called **out-performance options**
- Consider an exchange **call** option maturing T periods from now which allows its holder to obtain 1 unit of risky asset #1 in return for one unit of risky asset #2.
- S_t . . . the price of the risky asset #1 at time t
- K_t . . . the price of the risky asset #1 at time t
- δ_S . . . the dividend yield of the risky asset #1
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- σ_S, σ_K . . . the volatilities of the risky assets #1 and #2, respectively
- ρ . . . the **correlation** between the two assets

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Exchange Options: The pricing formula

- $$C(S, K, (\sigma_S, \sigma_K), r, T, (\delta_S, \delta_K)) = Se^{-\delta_S T} N(d_1) - Ke^{-\delta_K T} N(d_2)$$

  where

  $$d_1 = \frac{1}{\sigma\sqrt{T}}\left[\ln\left(\frac{Se^{-\delta_S T}}{Ke^{-\delta_K T}}\right) + \frac{1}{2}\sigma^2 T\right]$$
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  $$\sigma^2 = \sigma_S^2 + \sigma_K^2 - 2\rho\sigma_S\sigma_K$$

- In words, σ is the **volatility of** $\ln(S/K)$ (over the life of the call)
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Exchange Options: Application to options on bonds

In our case, the two risky assets are

S The bond

K The strike - Note that the strike should not be seen as constant. Its time-value (in the long run) is dependent on the interest rate which is not even deterministic!

- $S_t \dots$ denotes the value at time t of the bond, i.e., it is the **prepaid forward price** of the bond
- $K_t \dots$ denotes the value at time t of the strike, i.e., it is the *prepaid forward price* of the strike whose **nominal** value at time T is K

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Forward Contracts Revisited

- Recall that a forward contract is an agreement to pay a specified **delivery price** K at a **delivery date** T in exchange for an asset
- Let the asset's price at time t be denoted by S_t.
- Then, we denote the T-forward price of this asset at time t by $F_{t,T}[S]$.
- It is **defined** as the value of the delivery price K which makes the forward contract have the **no-arbitrage** price at time t equal to zero.

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Forward Contracts: Connection with bond-prices

- **Theorem:** Assume that zero-coupon bonds of all maturities are/can be traded. Then,

$$F_{t,T}[S] = \frac{S_t}{P(t,T)}, \text{ for } 0 \leq t \leq T$$

- **The argument:**

  Suppose that at time t you:

1. **Sell** the above forward contract - this is not a "real sale" as no income can be generated in doing so (by definition)
2. Also, you **short** $\frac{S_t}{P(t,T)}$ zero-coupon bonds - doing so you get the income of S_t
3. With the above produced income S_t, you by one share of the asset S

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Forward Contracts: Connection with bond-prices (cont'd)

You do nothing until time T

Then, at time T you:

1. **Deliver** the one share of asset S that you own
2. **Get** the delivery price K in return
3. **Cover** the short bond position - recall that the bonds we shorted all had maturity T at which time they are worth exactly \$1, i.e., the amount of the payment they produce at maturity

- The net-effect at time T is that you have $K - \frac{S_t}{B(t,T)}$ - this value must be equal to zero, or else there is arbitrage

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Forward Contracts on Bonds

- In particular, if the asset S is actually another bond with maturity $T+s$, we have that

$$F_{t,T}[P(T,T+s)] = \frac{P(t,T+s)}{P(t,T)}, \text{ for } 0 \le t \le T,\ s \ge 0$$

- So, the prepaid forward price at time t on the bond is $S_t = F_{t,T}[P(T,T+s)]P(t,T) = P(t,T+s)$ in the exchange option setting
- And, if the asset is just some nominal value given at time T, we can see this as K bonds which deliver \$1 at maturity T
- So, $K_t = KP(t,T)$ is the prepaid forward price we will use in the exchange option pricing formula
- The volatility that enters the pricing formula for exchange options is:

$$\begin{aligned} Var\left[\ln\left(\frac{S_t}{K_t}\right)\right] &= Var\left[\ln\left(\frac{P(t,T+s)}{KP(t,T)}\right)\right] \\ &= Var\left[F_{t,T}[P(T,T+s)]\right] \end{aligned}$$

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- So, the prepaid forward price at time t on the bond is $S_t = F_{t,T}[P(T,T+s)]P(t,T) = P(t,T+s)$ in the exchange option setting
- And, if the asset is just some nominal value given at time T, we can see this as K bonds which deliver \$1 at maturity T
- So, $K_t = KP(t,T)$ is the prepaid forward price we will use in the exchange option pricing formula
- The volatility that enters the pricing formula for exchange options is:

$$\begin{aligned} Var\left[\ln\left(\frac{S_t}{K_t}\right)\right] &= Var\left[\ln\left(\frac{P(t,T+s)}{KP(t,T)}\right)\right] \\ &= Var\left[F_{t,T}[P(T,T+s)]\right] \end{aligned}$$

Black formula

- If we assume that the bond forward price process $\{F_{t,T}[P(T, T+s)]\}_t$ agrees with the Black-Scholes assumptions and that its constant volatility is σ, we obtain the Black formula for a bond option:

$$C[F, P(0,T), \sigma, T] = P(0,T)[FN(d_1) - KN(d_2)]$$

where

$$d_1 = \frac{1}{\sigma\sqrt{T}}\left[\ln(F/K) + \frac{1}{2}\sigma^2 T\right]$$
$$d_2 = d_1 - \sigma\sqrt{T}$$

with $F = F_{0,T}[P(T, T+s)]$

Forward (Implied) Interest Rate

- We are now at time 0.
  Assume that you would like to earn at the interest rate in the period between time T and time $T + s$. Denote this forward interest rate by $R_0(T, T + s)$.
- The unit investment in the interest rate at time T until time $T + s$ should be consistent (in the sense of no-arbitrage) with the strategy that includes a zero-coupon bond maturing at time T and another with maturity at time $T + s$, i.e., we should have

$$1 + R_0(T, T + s) = \frac{P(0, T)}{P(0, T + s)}$$

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Forward Rate Agreements

- Consider a borrower (or, analogously, a lender) who wants to hedge against increases in the cost of borrowing a certain amount of money at a future date (that is, in the interest rate)
- Forward rate agreements (FRAs) are over-the-counter contracts that guarantee a borrowing or lending rate on a given principal amount
- FRAs are, thus, a type of forward contracts based on the interest rate:
  If the reference ("real") interest rate is above the rate agreed upon in the FRA, then the borrower **gets paid**.
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Forward Rate Agreements: Settlement time

- FRAs can be settled **at maturity** (in arrears), i.e., at the time the loan is repaid or **at the initiation** of the borrowing or lending transaction, i.e., at the time the loan is taken
- Let r denote the reference interest rate for the prescribed loan period
- If in arrears, then the payment is

$$(r - r_{FRA}) \times \text{notional principal}$$

- If at the initiation, then the payment is

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- One way to hedge the risk coming from the changes in the interest rate is, then, a simple FRA
- If the settlement is conducted in arrears, the payoff at time $T+s$ is

$$\text{spot } s\text{-period rate} - \text{forward rate} = R_T(T, T+s) - R_0(T, T+s)$$

- Alternatively, one (in this case a borrower) might buy a **call option on the FRA** - this should be a better strategy as there is no downfall in dropping interest rates

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- **Options on the FRA** are called caplets
- This option, at time $T + s$ pays

$$(R_T(T, T+s) - K_R)^+$$

where K_R denotes the strike

- If settled at time T, then the above type of option has payoff

$$\frac{1}{1 + R_T(T, T+s)}(R_T(T, T+s) - K_R)^+$$

due to "discounting"

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Forward Rate Agreements: Pricing caplets through the Black formula

- Using simple algebra, we can transform the above payoff into

$$(1 + K_R)\left(\frac{1}{1 + K_R} - \frac{1}{1 + R_T(T, T+s)}\right)^+$$

- Recalling the consistency equation

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we see that the value $\frac{1}{1+R_T(T,T+s)}$ is the value at time t of a zero-coupon bond paying $1 at time $T + s$

- Setting the value $\frac{1}{1+K_R}$ as the new strike, we see that we can use the Black formula to price the above described caplet

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- An interest rate **cap** is a collection of caplets
- Suppose a borrower has a floating rate loan with interest payments at times t_i, $i = 1, ..., n$. A cap would make the series of payments at times t_{i+1} given by

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