

A Binomial Interest Rate Model and the Black-Derman-Toy Model

- 1 A Binomial Interest Rate Model
 - Zero-coupon Bond Prices
 - Yields and Expected Interest Rates
 - Option Pricing
- 2 Black-Derman-Toy

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A Binomial Interest Rate Model: Notation

- h is the length of the binomial period; if it is not stated otherwise, we take that a period is 1 year, i.e., $h = 1$
- $r_{t_0}(t, T)$ is the **forward interest rate** at time t_0 for time t to time T
- $r_{t_0}(t, T; j)$ is the interest rate from t to T , where the rate is quoted at time $t_0 < t$ and the **state** (i.e., the **height** of the corresponding node in the binomial tree) is j
- At any time t_0 , there is a set of both spot (for $t = t_0$) and implied (for $t > t_0$) forward zero-coupon bond prices: $P_{t_0}(t, T; j)$
- p is the **risk-neutral** probability of an up move
- Note: Study the binomial tree in Figure 24.2 in the book. Do not get confused: This is indeed a schematic of the **three-period** interest rate model - the 1-yr rate is observable today; then, the 1-yr rate can go up or down over the first year (thus generating different 2-yr rates); again, there can be up or down movement during the second year (resulting in a set of possible 3-yr rates)

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Zero-Coupon Bond Prices

- Because the tree can be used at any node to value zero-coupon bonds of any maturity, the tree also generates implied forward interest rates of all maturities, as well as volatilities of implied forward rates
- Thus, we can equivalently specify a binomial interest rate tree in terms of any of the following:
 1. Interest rates
 2. Zero-coupon bond prices
 3. Volatilities of implied forward interest rates

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Zero-Coupon Bond Prices: The mechanics of the pricing procedure

- Each path in the binomial tree implies a realized discount factor \Rightarrow We can value a bond by considering separately each path the interest rate can take
- We then compute the expected discount factor, using risk-neutral probabilities
- Effectively, to evaluate the price of the zero-coupon bond, we perform a calculation of the following form:

$$\mathbb{E}^* \left[e^{-\sum_{i=0}^n r_i h} \right]$$

where n is the number of time-steps, r_i are the time- i rates from the binomial tree and \mathbb{E}^* is the expectation with respect to the risk-neutral measure (as generated by the risk-neutral probabilities p)

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Zero-Coupon Bond Prices: On the volatility

The volatility of the bond price is different from the behavior of a stock

- With a stock, uncertainty about the future stock price increases with horizon
- With a bond, volatility of the bond price initially grows with time. However, as the bond approaches maturity, volatility declines because of the boundary condition that the bond price at maturity is equal to the amount issued from the bond, i.e., \$1

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Why not use the average interest rate?

- Uncertainty causes bond yields to be lower than the expected average interest rate
- The discrepancy between yields and average interest rates increases with volatility
- Therefore, we cannot price a bond by using the expected interest rate

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The Binomial Tree

Using the binomial tree to price a bond option works the same way as bond pricing

- Suppose we have a call option with strike price K on a $T - t$ -year zero-coupon bond, with the option expiring in $t - t_0$ periods.
- The expiration value of the option is

$$O(t, j) = [P_t(t, T; j) - K]^+$$

- To price the option we work **recursively** backward through the tree using risk-neutral pricing, as with an option on a stock. This strategy tells us that the value one period earlier and at the node at height j is

$$O(t - h, j) = P_{t-h}(t - h, t; j) [p O(t, 2j + 1) + (1 - p) O(t, 2j)]$$

- In the same way, we can value an option on a yield, r an option on any instrument that is a function of the interest rate

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Delta-Hedging

- Delta-hedging works for the bond option just as for a stock option
- The underlying asset is a zero-coupon bond maturing at T , since that will be a $T - t$ -period bond in period t
- Each period, the delta-hedged portfolio of the option and underlying asset is financed by the short-term bond, paying whatever one-period interest rate prevails at that node

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Calibration

- The models we have examined are arbitrage-free in a world consistent with their assumptions.
- In the real world, however, they will generate **apparent** arbitrage opportunities, i.e., observed prices will not match theoretical prices
- This is not surprising:
Recall the smooth yield curves obtained in the Vasicek and CIR models - this is not going to be consistent with the observed evolution
- Matching a model to fit the data is called **calibration**
- This calibration ensures that it matches observed yields and volatilities, but **not** necessarily the evolution of the yield curve
- The Black-Derman-Toy (BDT) tree is a binomial interest rate tree calibrated to match zero-coupon yields and a particular set of volatilities

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The Black-Derman-Toy Tree

- The basic idea of the BDT model is to compute a binomial tree of short-term interest rates, with a flexible enough structure to match the data
- Consider market information about bonds that we would like to match; namely, we would like to match (effective annual) yield to maturity, bond prices and the volatility of the bond **yields** (see Table 24.2 in the textbook)
- Black, Derman, and Toy describe their tree as driven by the short-term rate, which they assume is **lognormally distributed**
- For each period in the tree, there are two parameters:
 1. R_{it} is a **rate level parameter** at a given time; and
 2. σ_i is a **volatility** parameter
- These parameters are used to match the tree with the data
- The general form of the BDT tree is displayed in Figure 24.4 in the textbook

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The Black-Derman-Toy Tree: Measuring volatilities

- Let the time- h price of a zero-coupon bond maturing at T when the time- h short-term rate is $r(h)$ be denoted by $P[h, T, r(h)]$
- Then, the **annualized** yield of the bond is

$$y[h, T, r(h)] = P[h, T, r(h)]^{-1/(T-h)} - 1$$

- Assume that at time h , the short-term rate can take on the two values: r_u and r_d . Then, the annualized lognormal yield volatility (variance of the Bernoulli r.v., really!) equals

$$\frac{1}{2\sqrt{h}} \ln \left(\frac{y(h, T, r_u)}{y(h, T, r_d)} \right)$$

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$$\frac{1}{2\sqrt{h}} \ln \left(\frac{y(h, T, r_u)}{y(h, T, r_d)} \right)$$

The Black-Derman-Toy Tree: Verification

- First, you enter the market data into the BDT tree - using the expressions at the nodes of the generic BDT tree and plugging in the data - which depicts the 1-year effective annual rates (see Figure 24.5)
- The tree will be different from the binomial trees we have seen so far, e.g., Unlike a stock-price tree, the nodes are not necessarily “centered” on the previous periods nodes; this is because the tree is matching the **data** by construction
- One can verify that the recipe for the values at the nodes given in the generic BDT tree is indeed consistent with the data:
 1. To verify that the tree matches the yield curve, one should compute the prices of zero-coupon bonds with maturities of 1, 2, 3, and 4 years
 2. To verify the volatilities, one should compute the prices of 1-, 2-, and 3-year zero-coupon bonds at year 1, and then compute the yield volatilities of those bonds
- You can read on the verification steps in the textbook as a reading assignment

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Constructing the Black-Derman-Toy Tree

- We start at early nodes and work to the later nodes, building the tree outward
- The **first node** is given by the prevailing 1-year rate R_0
- Let the 1-year bond price be denoted by P_0 ; it is given in the table and it must be that

$$P_0 = 1/(1 + R_0)$$

- Whence, we solve for R_0 and write it in the tree

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Constructing the Black-Derman-Toy Tree (cont'd)

- For the **second** node, the year-1 price of a 1-year bond is denoted by $P(1, 2, r_u)$ or $P(1, 2, r_d)$, depending on the movement of the interest rate.
- Let P_1 be the observed year-1 price of a 1-year bond; we require that two consistency conditions be satisfied:

$$\begin{aligned}P_1 &= \frac{1}{1 + R_0} \left[\frac{1}{2} P(1, 2, r_u) + \frac{1}{2} P(1, 2, r_d) \right] \\&= \frac{1}{1 + R_0} \left[\frac{1}{2(1 + R_1 e^{2\sigma_1})} + \frac{1}{2(1 + R_1)} \right] \\R_0 &= \frac{1}{2} \ln \left(\frac{R_1 e^{2\sigma_1}}{R_1} \right)\end{aligned}$$

- This is a system of two equations with two unknowns R_1 and σ_1 - you solve for the two and follow the recipe from the generic BDT tree to enter values at the nodes
- In the same way, it is possible to solve for the parameters for each subsequent period

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Black-Derman-Toy Examples: Caplets and caps

- Recall that an interest rate cap pays the difference between the realized interest rate in a period and the interest cap rate, if the difference is positive
- The payments in each node in a tree are the present value of the cap payments for the interest rate at that node

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