Equations of Value

1. Single Deposit Under Compound Interest

2. Investments with Multiple Contributions

The amount function

- Let the principal be denoted by C. Then, in the usual notation, the amount function reads as

$$A_C(T) = C(1+i)^T \text{ for every } T \geq 0$$

- The above equation is referred to as the T time equation of value
- Knowing any three among the values $A_C(T), C, T, i$ allows us to calculate the fourth one
- We will illustrate this fact through examples

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Unknown principal

- On July 1, 2024, you will need $\$1,000$. Assume that the annual effective interest rate equals 0.0625. How much would you have to deposit on October 1, 2020 in order to be able to withdraw the $\$1,000$ from the account on July 1, 2024?

$\Rightarrow$ The start of the loan term is October 1, 2020. So, we call that time zero. Hence, the time at which we are given the accumulated amount we want to produce is $T = 3.75$ (in years).

The equation of value reads as

$$1000 = C \cdot (1 + 0.0625)^{3.75}$$

Solving for C, we obtain $C = 796.65$.

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Unknown length of investment

- Find the length of time necessary for \$1000 to accumulate to \$1500 if invested at 6% per annum compounded semiannually

⇒ It is more convenient to think of time-units in this situation as half-years. Let us denote the unknown number of half-years by n.

Then the equation of value reads as

$$1000\left(1+\frac{0.06}{2}\right)^n = 1500$$

We get $n \approx 13.7$, i.e., the sought for length of time is about 6.85 years.

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Unknown time: Doubling your money

- At a nominal interest rate of 10% convertible quarterly, how long will it take you to double your money?

⇒ Because the wealth accumulated at any given time is proportional to the principal, we can assume (without loss of generality) that the initial investment is a single dollar.

Then the equation of value becomes

$$\left(1 + \frac{0.04}{4}\right)^n = 2$$

where n denotes the unknown length of time (in quarter-years) that it takes to double the initial investment

Solving for n, we get $n \approx 28.071$ quarters of a year, i.e., the length of time we were looking for is about 7 years.

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Unknown interest rate

- At what interest rate convertible quarterly would \$1000 accumulate to \$1600 in six years?

⇒ We need to find $i^{(4)}$ from the equation of value

$$1000\left(1+\frac{i^{(4)}}{4}\right)^{24}=1600$$

We get

$$\frac{i^{(4)}}{4}=(1.6)^{1/24}-1=0.019776$$

and so $i^{(4)}=0.0791$.

- *Assignment:* Problems 2.2.1,2,3,6

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The time τ equation of value

- Consider a sequence of contributions to an investment account made at times $\{t_k\}$ and represented by values $\{C_{t_k}\}$
- This sequence of investments is liquidated at a time T and the resulting balance is denoted by B
- Suppose that a certain intermediate time τ bears a significance for the investor and that the investor wants to **value** all the contributions at that specific time τ
- Then the value of all the combined contributions to the account valued at time τ must be equal to the final balance B valued at time τ
- Formally, we get the time τ equation of value

$$\sum_k C_{t_k} \frac{a(\tau)}{a(t_k)} = B \frac{a(\tau)}{a(T)}$$

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- In the case that we choose $\tau = 0$ the time 0 equation of value becomes

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- In words, it suffices to discount all the contributions and the balance from the times their corresponding cash flows took place to the initial time 0

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An example: Unknown payment

- In return for a promise to receive \$600 at the end of 8 years, Roger agrees to pay \$100 at once, \$200 at the end of 5 years, and to make a further payment X at the end of 10 years.

  Assume that the nominal rate of interest is 8% convertible semiannually.

  Find the payment at the end of 10 years.

⇒ Again, let us count time-periods in half-years.

  Note that there is no particular liquidation moment; this means that there is no remaining balance after all the cash flows outlined in the problem take place; we could choose any moment later than time 10 and say that $B = 0$

  Since we are looking for a payment that is made at time $\tau = 10$, let us make that point in time our reference point. The equation of value is

$$100(1.04)^{20} + 200(1.04)^{10} - 600(1.04)^{4} + X = 0$$

- We solve the above equation and get $X = 186.76$.

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An example: Unknown rate of interest

- At what effective rate of interest will the present value of the investment consisting of \$2000 deposited at the end of 2 years and \$3000 deposited at the end of four years be equal to \$4000?

⇒ The above wording implies that the reference point for our calculations should be time 0. Since this means that we are discounting, it is more convenient to write the equation of value in terms of the discount factor v.
The time 0 equation of value is

$$2000v^2 + 3000v^4 = 4000$$

We get a quadratic in v^2:

$$3v^4 + 2v^2 - 4 = 0$$

which yields only one positive solution $v^2 = 0.8685$
So,

$$(1 + i)^2 = v^{-2} = 1.1514$$

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- The present value of two payments of \$100 each to be made at the end of n years and $2n$ years is \$100. If $i = 0.08$, find n.

$\Rightarrow$ The equation of value (at time 0) is

$$100(1+0.08)^{n} + 100(1+0.08)^{2n} = 100,$$

i.e.,

$$(1+0.08)^{-n} + \left((1+0.08)^{-n}\right)^{2} = 1$$

Set $A = (1.08)^{-n}$; the above equation becomes

$$A^2 + A - 1 = 0$$

We only keep the positive solution $A = \frac{1}{2}(\sqrt{5} - 1)$
and from there we get that $n = -\ln(A)/\ln(1.08) \approx 6.25$ years

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$$(1 + 0.08)^{-n} + \left((1 + 0.08)^{-n}\right)^2 = 1$$

Set $A = (1.08)^{-n}$; the above equation becomes

$$A^2 + A - 1 = 0$$

We only keep the positive solution $A = \frac{1}{2}(\sqrt{5} - 1)$
and from there we get that $n = -\ln(A)/\ln(1.08) \approx 6.25$ years

Method of Equated Time: Exact approach

- The type of problem we wish to consider next is the following:

  An investor makes the contributions $\{C_{t_k}; k = 1, 2, \ldots n\}$ at times $\{t_k\}$. Find the time T so that a single payment of $C = \sum_{k=1}^{n} C_{t_k}$ at time T has the same value at time 0 as the n contributions.

- First we formalize the conditions in the above problem:

$$Cv^T = \sum_{k=1}^{n} C_{t_k} v^{t_k}$$

- Hence,

$$v^T = \frac{1}{C} \sum_{k=1}^{n} C_{t_k} v^{t_k}$$

and so

$$T = \frac{1}{\ln(v)} \cdot \ln\left(\frac{1}{C} \sum_{k=1}^{n} C_{t_k} v^{t_k}\right)$$

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Method of Equated Time: The approximation

- An approximation to the above solution is

$$\bar{T} = \frac{1}{C}\sum_{k=1}^{n} C_{t_k} t_k = \sum_{k=1}^{n} \frac{C_{t_k}}{C}\, t_k$$

- The above weighted average of payment times is called the method is equated time approximation
- It is worth noticing that we consistently have

$$\bar{T} \geq T$$

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An Example

- Payments of \$100, \$200 and \$500 are due at the ends of years $2, 3$ and 8, respectively. Assuming an effective interest rate of 5%, find the time t^* at which a single payment of \$800 would be equivalent to the above combination of payments.

⇒ Using the method of equated time, we get an approximation to the time t^*, i.e.,

$$\bar{t}^* = \frac{100}{800} \cdot 2 + \frac{200}{800} \cdot 3 + \frac{500}{800} \cdot 8 = 6 \text{ years}$$

Our exact model yields the following sequence of calculations: The present values of the three payments are

$$100 \cdot 0.90703,\ 200 \cdot 0.86384,\ 500 \cdot 0.67684$$

So,

$$\frac{1}{800}(100 \cdot 0.90703 + 200 \cdot 0.86384 + 500 \cdot 0.67684) = 0.75236$$

Finally,

$$t^* = -\frac{\ln(0.75236)}{\ln(1.05)} = 5.83 \text{ years}$$

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Assignments

- *Assignment:* See Example 2.3.6 for a problem which is not to easy to tackle without a calculator
  Problems 2.3.4,5, 9, 10