Yield Rates

1 Investment Return

2 Reinvestment Considerations

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- At an unknown interest rate i, an investment of 1000 immediately and 1500 at the end of the second year accumulates to 2600 at the end of the fourth year. Find i.
- \Rightarrow Set j=1+i; then, the equation of value at time au=4 reads as

$$1000j^4 + 1500j^2 = 2600$$

Solving the quadratic, we get $j^2 = 1.028342$. So, i = 0.014.

 The rate we have just calculated is know as the yield rate for an investment.

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$$\sum_{k=1}^{n} C_{t_k} (1+i)^{\tau-t_k} = B(1+i)^{\tau-T}$$

- The rate of interest i which satisfies the above equation (for all other ingredients given and fixed) is called the (annual) yield rate or internal rate of return for that investment
- The yield rate is often understood as a measure of quality of a certain investment.
- There is a slight problem with this line of thinking, as we can see from the following example . . .

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- Consider a transaction in which Roger makes payments of \$100 immediately and \$132 at the end of the two years in exchange for a payment in return of \$230 at the end of one year. What is the yield rate of this investment scheme?
- ⇒ The equation of value is

$$100(1+i)^2 + 132 = 230(1+i)$$

which yields the quadratic equation

$$(1+i)^2 - 2.3(1+i) + 1.32 = 0$$

$$[(1+i)-1.1] \cdot [(1+i)-1.2] = 0$$

- So that both i = 0.1 and i = 0.2 may be considered as solutions!
- We cannot have any preference towards a particular choice of the yield rate above, but - we have just illustrated why it is of interest to show that a yield rate is *unique* in a particular situation . . .

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- Assume that Roger is able to borrow \$100 from a bank for one year at 8% effective and immediately lend the said \$100 to Harry for one year at 10% effective. What is Roger's yield rate for this combination of transactions?
- ⇒ Apparently, Roger is able to make a \$2 profit at the end of the one year - without any net investment at time zero. We might say that this means that the yield rate is infinite ...
 - However, if Roger were able to borrow \$1000 and then lend that amount of money to Harry (both at the same effective interest rates), his profit at the end of the one year would be \$20; this is clearly preferable to the \$2 profit that he generated in the first case and, yet, the yield rate would be (by the same reasoning as above) infinite.
 - It is sensible to wish to be able to compare the above two cases, but this comparison is not allowed through the definition of yield rates ...
- We choose the say that the yield rate is undefined in situations such as the ones above . . .

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- What is the yield rate on a transaction in which a person makes a payment of \$100 immediately and \$101 at the end of two years, in exchange for a payment of \$200 at the end of one year?
- \Rightarrow We need to find *i* from the equation of value

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We get

$$100i^2 = -1$$

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A sufficient condition for uniqueness of Yield rates

- Hopefully, we have managed to illustrate the importance of showing the **existence and uniqueness** of yield rates . . .
- An easily verified condition for uniqueness of yield rates is the following:

Suppose that the contributions take place at times $t_1 < t_2 < \dots < t_n$ and that there **exists** a yield rate i > -1 such that the outstanding balances

$$B_{t_k}(i) = \sum_{i=1}^k C_{t_i} (1+i)^{t_k-t_i}, \text{ for } k \le n-1$$

are all **of the same sign**. Then i is the **unique** vield rate greater than -1.

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Then i is the unique yield rate greater than -1.

- Roger's friend Harry wants to open a pizza parlor. So, Roger lends him \$50,000. Harry should return \$55,000 in one year. However, in a year Casper - another friend of Roger's - decides to enter a business venture himself and seeks financial assistance from Roger.
- Then, Roger asks Harry to give Casper \$40,000 and gets the remaining \$15,000.
- In another year, Casper pays Roger \$45,000. What is Roger's yield rate in this arrangement?
- We are asked only about Roger's yield rate. So, we are going to only look at transactions from his perspective - ignoring all the cash flows that occur between Harry and Casper.

Then the time 2 equation of value, when we concentrate on Roger becomes

$$50000(1+i)^2 = 15000(1+i)^1 + 45000$$

and so $i = \frac{1}{20}(3 + \sqrt{369}) - 1$ (where we kept only the positive yield rate)

• We used the "bottom line approach" - focusing on a single investor and only on the transactions that involve that investor

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- Assume that Roger deposits \$100 at the beginning of each year for 3 years to an account with the effective annual rate of 5%.
 He also seizes the opportunity to invest the *interest payments* from that account at an annual effective rate of 10%. Find the accumulated value of all Roger's accounts a year after his last deposit.
- ⇒ A time line is really important here! Altogether, Roger ends up with

$$100 + 100 + 100 = 300$$

on his primary account

On the other hand, his interest payments from his primary account had the nominal values of \$5,\$10 and \$15 and have grown to

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Assignments

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 Problems 2.4. 1,2,5,6,8,9
 Problems 2.5. 1.2.3
- For fun: Look for "Descartes' rule of signs" on Wikipedia; it is a simple rule that can occasionally help you verify that there is a unique yield rate in a given problem ...

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