More on Yield Rates

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The Set-Up

- Consider an arbitrary loan term and choose the unit length of time to be exactly the length of time of that term (e.g., 3 months, 2.5 years, 189 days, ...), i.e., we set $T = 1$
- A ... denotes the **initial** amount of money on the account, i.e., the amount of money on the account at time 0
- C ... denotes the net contributions to the fund, i.e.,

$$C = \sum_{0<t<1} C_t$$

- I ... denotes the interest earned by the fund during the time interval $(0, 1)$
- B ... denotes the final amount of money on the account, i.e., the balance on the account at time 1
- Necessarily,

$$B = A + C + I$$

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Introducing the interest rate to the set-up

- j ... denotes the interest rate **for the period** $[0,1]$
- Thus,

$$I = A \cdot j + \sum_{0<t<1} C_t \left[(1+j)^{1-t} - 1\right]$$

- It may be difficult to evaluate the exponential function in the brackets above; so, we **approximate** the compound interest by simple interest at the same rate j, i.e., we notice that

$$C_t \left[(1+j)^{1-t} - 1\right] \approx C_t \cdot j(1-t)$$

- Note that this approach agrees with the Taylor expansion of the exponential function ...
- Combining the above two displays, we can approximate the earned interest I by

$$I \approx A \cdot j + \sum_{0<t<1} C_t \cdot j(1-t)$$

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An approximation for the rate of return

- Equivalently, we can write

$$j \approx \frac{I}{A + \sum_{0<t<1} C_t \cdot (1-t)}$$

- Additionally, under the assumption that we approximate all times at which an actual contribution takes place by a single constant time k, the above approximation becomes even simpler:

$$j \approx \frac{I}{A + \sum_{0<t<1} C_t \cdot (1-k)} = \frac{I}{A + C \cdot (1-k)}$$

where C denotes the total sum of all the contributions made during the interval $(0, 1)$

- Recalling the "law of preservation of money" equality

$$B = A + C + I$$

we conclude that

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An coarser approximation for the rate of return

- If we choose $k = 1/2$, we can further simplify the above approximation to

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An Example

- At the beginning of the year an investment fund was established with an initial deposit of \$1,000. Another deposit of \$500 was made at the end of four months.
- Withdrawals of \$200 and \$100 were made at the end of six months and eight months, respectively.
- The amount in the fund at the end of the year is \$1272. Find the approximate effective rate of interest earned by the fund during the year, approximating the compound interest with the simple interest.

⇒ In the notation of this section,

$$A = 1000,\ C_{4/12} = 500,\ C_{6/12} = -200,\ C_{8/12} = -100,\ B = 1272.$$

So, $I = 1272 - (1000 + 500 - 200 - 100) = 72$.
Then,

$$j \approx \frac{72}{1000 + \frac{2}{3} \cdot 500 - \frac{1}{2} \cdot 200 - \frac{1}{3} \cdot 100} = 0.06$$

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An Example: What if you are not given the times of contributions?

- Find the effective rate of interest earned during a calendar year by a company with the following data:
  * Assets, beginning of year . . . 10,000,000
  * Premium income . . . 1,000,000
  * Gross investment income . . . 530,000
  * Policy benefits . . . 420,000
  * Investment expenses . . . 20,000
  * Other expenses . . . 180,000

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An Example: What if you are not given the times of contributions? (cont'd)

⇒ We have

$$
\begin{aligned}
A &= 10,000,000 \\
B &= 10,000,000 + 1,000,000 + 530,000 \\
&\quad - 420,000 - 20,000 - 180,000 \\
&= 10,910,000 \\
I &= 530,000 - 20,000 = 510,000.
\end{aligned}
$$

We are given only to cumulative contributions for the year, so our only choice is to use the approximation $k = 1/2$:

$$j \approx \frac{2 \cdot 510,000}{10,000,000 + 10,910,000 - 510,000} = 0.05.$$

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An Example: FM Sample Problem #5

- An association had a fund balance of 75 on January 1^{st} and 60 on December 31^{st}.
- At the end of every month during that year, the association deposited 10 from membership fees.
- There were withdrawals of 5 on February 28^{th}, 25 on June 30^{th}, 80 on October 15^{th}, and 35 on October 31^{st}.
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⇒ Caveat: What they really have in mind is the **approximate** dollar-weighted rate of return, assuming simple interest throughout the year

In the notation of this section, we have

$$A = 75, \text{ initial amount on the account}$$
$$B = 60, \text{ balance} = \text{final amount on the account}$$
$$C = 120 - 145, \text{ contributions from membership offset by withdrawals}$$
$$I = 60 + 145 - 120 - 75 = 10, \text{ interest}$$

Thus,

$$j \approx \frac{10}{75 + \sum_{k=0}^{11} \frac{k}{12} \cdot 10 - \frac{10}{12} \cdot 5 - \frac{6}{12} \cdot 25 - \frac{2.5}{12} \cdot 80 - \frac{2}{12} \cdot 35}$$

So, $j \approx 0.11$

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