Continuously paying annuities

1. Compound interest

2. Compound interest: Increasing payments

3. General Accumulation Function

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The Set-up: Level annuity

Assume that we have compound interest with the effective interest rate per interest period equal to i.
Consider the following continuous annuity:

- the annuity lasts for n interest periods;
- the payments take place continuously, at a rate of 1 per interest period.
- $\bar{a}_{\overline{n}|\,i}$ ... stands for the present value of the above annuity, i.e.,

$$\bar{a}_{\overline{n}|\,i} = \frac{1 - e^{-\delta n}}{\delta}$$

- $\bar{s}_{\overline{n}|\,i}$ ... stands for the accumulated value of the above annuity, i.e.,

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An Example

- Find the constant force of interest δ at which

$$\bar{s}_{\overline{20}|} = 3 \cdot \bar{s}_{\overline{10}|}$$

⇒ Using the formula above, we get that

$$\frac{e^{20\cdot\delta} - 1}{\delta} = 3 \cdot \frac{e^{10\cdot\delta} - 1}{\delta}$$

So,

$$e^{20\delta} - 3 \cdot e^{10\delta} + 2 = 0$$

Hence,

$$(e^{10\delta} - 1)(e^{10\delta} - 2) = 0$$

We reject the root $\delta = 0$ and obtain the unique solution

$$\delta = \frac{\ln(2)}{10} = 0.0693$$

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The Set-up: Unit increase in payments

Assume that we have compound interest with the effective interest rate per interest period equal to i.

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- the annuity lasts for n interest periods;
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$$(\bar{I}\bar{a})_{\overline{n|}\, i} = \frac{\bar{a}_{\overline{n|}\, i} - nv^n}{\delta}$$

- It is easy to see what happens by noting that

$$(\bar{I}\bar{a})_{\overline{n|}\, i} = \int_0^n t \cdot v^t \, dt$$

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Unit payment stream

- Let $v(t)$ denote the general discount function
- Let us first consider the basic continuous annuity, i.e., the annuity that pays at the unit rate at all times.
- Then, the present value of such an annuity with length n equals

$$\int_0^n v(t)\,dt$$

- We still denote the above present value by $\bar{a}_{\overline{n}|}$
- In the special case of compound interest, the above formula collapses to the one already familiar to us from the compound interest set-up You can verify this through simple integration ...

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Any payment stream

- Let $f(t)$ be a continuous function which represents the rate of payments of a continuous annuity on the time interval $[0, n]$
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