Convexity

Modified Convexity

- Recall the second-order Taylor approximation for the relative change in the price P as a function of the interest rate i

$$\frac{P(i) - P(i_0)}{P(i_0)} \approx \frac{P'(i_0)}{P(i_0)}(i - i_0) + \frac{1}{2} \cdot \frac{P''(i_0)}{P(i_0)}(i - i_0)^2$$

- Guided by the above approximation, we define the modified convexity by

$$C(i_0, 1) = \frac{P''(i_0)}{P(i_0)}$$

- Now, the above approximation reads as

$$\frac{P(i) - P(i_0)}{P(i_0)} \approx -D(i_0, 1)(i - i_0) + \frac{1}{2} \cdot C(i_0, 1)(i - i_0)^2$$

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Modified Convexity for $i^{(m)}$; Macaulay Convexity

- Similarly to the duration discussion, we define:

$$C(i,m) = \frac{\frac{d^2}{d(i^{(m)})^2} P(i)}{P(i)}$$

$$C(i,\infty) = \frac{\frac{d^2}{d\delta^2} P(i)}{P(i)}$$

- $C(i,\infty)$ is called the Macaulay convexity, and can be expressed explicitly as

$$C(i,\infty) = \sum_{t\geq 0} \left(\frac{C_t(1+i)^{-t}}{P(i)} \right) t^2$$

- Again, it is a **weighted average**

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An Equality

$$\left(1+\frac{i^{(m)}}{m}\right)^{2} C(i,m) = C(i,\infty) + \frac{1}{m} D(i,\infty)$$

Sufficient condition for **decrease** of $D(i, \infty)$

- Assume that all the cashflows are nonnegative and that there is more than one instant at which the cashflow is positive, then $D(i, \infty)$ is **decreasing** in i

Dispersion

- Dispersion is a measure of how spread out the payment times are around the Macaulay duration $D(i, \infty)$.
- Formally, dispersion is defined as

$$V(i) = \frac{1}{P(i)} \sum_{t \geq 0} (t - D(i, \infty))^2 C_t \, e^{-\delta \, t}$$

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