Immunization

Surplus

Note: We still assume that the yield curve is flat and that its shifts are parallel . . .

- A set of cashflows consists of:

A assets: $A_t \geq 0, t \geq 0$

L liabilities: $L_t \geq 0, t \geq 0$

- Then, the surplus (valued at the interest rate i) is defined as

$$S(i) = \sum_{t \geq 0} (A_t - L_t)(1+i)^{-t} = \sum_{t \geq 0} A_t (1+i)^{-t} - \sum_{t \geq 0} L_t (1+i)^{-t}$$

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Redington Immunization

- Denote the current interest rate by i_0

Cond Assume that

$$S(i_0) = 0, \quad S'(i_0) = 0, \quad S''(i_0) > 0$$

- Then, $S(i_0 + h) \geq 0$ for all sufficiently small h
- Moreover, condition **Cond** is equivalent to the following:

PV The present values of assets and liabilities are equal, **AND**

MacD The Macaulay durations of assets and liabilities are equal, **AND**

MacC The Macaulay convexities of assets are **at least as large** as those of liabilities

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An Example

Roger owes Harry \$1,100 to be repaid at the end of one year. He is required to set up an investment fund to meet this obligation. The only investments available are:

MM a money-market fund currently earning 10% (with daily updates on the changes in this interest rate), and

B two-year, zero-coupon bonds earning 10%.

⇒ Let us denote by X the amount invested in the money market, and by Y the amount invested in two-year, zero-coupon bonds.

Then, the cashflows correspodning to these two investments are Roger's assets, while his debt to Harry constitutes a liability.

If the underlying interest rate is denoted by i, then

$$S(i) = X + 1.21Y(1+i)^{-2} - 1100(1+i)^{-1}$$
$$S'(i) = -2.42Y(1+i)^{-3} + 1100(1+i)^{-2}$$
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An Example (cont'd)

In particular, at the current interest rate $i_0 = 0.1$, we have that

$$S(i_0) = X + Y - 1000$$
$$S'(i_0) = -\frac{2Y}{1.1} + \frac{1000}{1.1}$$
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A Redington Immunization strategy would require Roger's investments X and Y to be such that the condition **Cond** is satisfied at the current interest rate. Then, small changes in the interest rate would still leave the surplus function nonnegative!!!

So, we need to set

$$S(i_0) = X + Y - 1000 = 0$$
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We get that $X = Y = 500$ is the unique solution to the above system of equations.

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An Example (cont'd, still)

If the condition on the second derivative in **Cond** is not satisfied for these values, then Redington immunization is not feasible in the given market.

However,

$$S''(i_0) = \frac{7.26 \cdot 500}{1.1^4} - \frac{2200}{1.1^3} \approx 826 > 0$$

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Now, we know that Roger's surplus function remains nonnegative for small shifts in the interest rate. But, we do not know what "small shifts" means in this particular situtation.

In this case, we can explicitly get that

$$S(i_0 + 0.01) = S(0.11) = 0.0406 > 0$$
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Finally, if all the conditions in **Cond** are verified as true, then we should have the consequences on the Macaulay duration and convexity of the assets and liabilities be true, as well...

For the assets above, we have that

$$P_A(i_0) = X + 1.21Y(1+i_0)^{-2} = 1,000$$
$$P'_A(i_0) = -2.42Y(1+i_0)^{-3} = -909.09$$

So,

$$D_A(i_0, 1) = -\frac{P'(i_0)}{P(i_0)} = 0.90909$$

On the other hand, we could decide to calculate the modified duration of all assets as the weighted average of the modified durations of the two investments in the money market and in the bond

The money market fund has the duration of 0, and the two-year zero-coupon bonds have the modified duration of 2/1.1

The weighted average of these values is

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An Example (final observation)

- For the convexity, we use

$$P''(i_0) = 7.26Y(1+i_0)^{-4} = 2479.34$$

- Then,

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- *Assignment:* Calculate the duration of the liabilities and verify the second consequence of **Cond**

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