Continuously paying annuities

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The Set-up: Level annuity

Assume that we have compound interest with the effective interest rate per interest period equal to i.
Consider the following continuous annuity:

- the annuity lasts for n interest periods;
- the payments take place continuously, at a rate of 1 per interest period.
- $\bar{a}_{\overline{n}|\,i}$ ...stands for the present value of the above annuity, i.e.,

$$\bar{a}_{\overline{n}|\,i} = \lim_{m\to\infty} a^{(m)}_{\overline{n}|\,i} = \frac{1-e^{-\delta n}}{\delta}$$

- $\bar{s}_{\overline{n}|\,i}$ ...stands for the accumulated value of the above annuity, i.e.,

$$\bar{s}_{\overline{n}|\,i} = \lim_{m\to\infty} s^{(m)}_{\overline{n}|\,i} = \frac{e^{\delta n}-1}{\delta}$$

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An Example

- Find the constant force of interest δ at which

$$\bar{s}_{\overline{20}|} = 3 \cdot \bar{s}_{\overline{10}|}$$

$\Rightarrow$ Using the formula above, we get that

$$\frac{e^{20\cdot\delta-1}}{\delta} = 3 \cdot \frac{e^{10\cdot\delta-1}}{\delta}$$

So,

$$e^{20\delta} - 3 \cdot e^{10\delta} + 2 = 0$$

Hence,

$$(e^{10\delta} - 1)(e^{10\delta} - 2) = 0$$

We reject the root $\delta = 0$ and obtain the unique solution

$$\delta = \frac{\ln(2)}{10} = 0.0693$$

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An Inequality

- We have that

$$\ddot{a}_{\overline{n}|} > \ddot{a}_{\overline{n}|}^{(m)} > \bar{a}_{\overline{n}|} > a_{\overline{n}|}^{(m)} > a_{\overline{n}|}$$