Duration

Assumptions

- Compound Interest
- Flat term structure of interest rates, i.e., the spot rates are all equal regardless of the term. So, the spot rate curve is flat.
- Parallel shifts in the term structure, i.e., as time goes by all the spot rates move together by the same amount. So, the spot rate curves are simply translated as the spot rates change with time at which they are observed.

Assumptions

- Compound Interest
- Flat term structure of interest rates, i.e., the spot rates are all equal regardless of the term. So, the spot rate curve is flat.
- Parallel shifts in the term structure, i.e., as time goes by all the spot rates move together by the same amount. So, the spot rate curves are simply translated as the spot rates change with time at which they are observed.

Assumptions

- Compound Interest
- Flat term structure of interest rates, i.e., the spot rates are all equal regardless of the term. So, the spot rate curve is flat.
- Parallel shifts in the term structure, i.e., as time goes by all the spot rates move together by the same amount. So, the spot rate curves are simply translated as the spot rates change with time at which they are observed.

The price curve

- Consider a set of future cash-flows $\{C_t : t \geq 0\}$, e.g., coupons and the redemption payment corresponing to a bond.
- If the spot rate is equal to i, this set of cash-flows has the present value

$$P(i) = \sum_{t \geq 0} C_t (1+i)^{-t}$$

- This present value can be understood as the price of the cash-flows (i.e., the investment that is equivalent to that set of cash-flows)
- If one draws a graph of the price P as a function of the interest rate, one gets a price curve which is typically convex, decreasing and has a horizontal asympote at zero as $i \to \infty$.

The price curve

- Consider a set of future cash-flows $\{C_t : t \geq 0\}$, e.g., coupons and the redemption payment corresponing to a bond.
- If the spot rate is equal to i, this set of cash-flows has the present value

$$P(i) = \sum_{t \geq 0} C_t (1+i)^{-t}$$

- This present value can be understood as the price of the cash-flows (i.e., the investment that is equivalent to that set of cash-flows)
- If one draws a graph of the price P as a function of the interest rate, one gets a price curve which is typically convex, decreasing and has a horizontal asympote at zero as $i \to \infty$.

The price curve

- Consider a set of future cash-flows $\{C_t : t \geq 0\}$, e.g., coupons and the redemption payment corresponing to a bond.
- If the spot rate is equal to i, this set of cash-flows has the present value

$$P(i) = \sum_{t \geq 0} C_t (1+i)^{-t}$$

- This present value can be understood as the price of the cash-flows (i.e., the investment that is equivalent to that set of cash-flows)
- If one draws a graph of the price P as a function of the interest rate, one gets a price curve which is typically convex, decreasing and has a horizontal asympote at zero as $i \to \infty$.

The price curve

- Consider a set of future cash-flows $\{C_t : t \geq 0\}$, e.g., coupons and the redemption payment corresponing to a bond.
- If the spot rate is equal to i, this set of cash-flows has the present value

$$P(i) = \sum_{t \geq 0} C_t (1+i)^{-t}$$

- This present value can be understood as the price of the cash-flows (i.e., the investment that is equivalent to that set of cash-flows)
- If one draws a graph of the price P as a function of the interest rate, one gets a price curve which is typically convex, decreasing and has a horizontal asympote at zero as $i \to \infty$.

The Taylor expansion of P

- Note that P is infinitely differentiable; as such it admits a Taylor expansion.
- So, we have the following second-Taylor-polynomial approximation for P around any i_0:

$$P(i) \approx P(i_0) + P'(i_0)(i - i_0) + \frac{1}{2} P''(i_0)(i - i_0)^2$$

  for i close to i_0
- The expressions for the derivatives are

$$P'(i) = -\sum_{t \geq 0} C_t \, t \, (1+i)^{-1-t}$$

$$P''(i) = \sum_{t \geq 0} C_t \, t \, (t+1) \, (1+i)^{-2-t}$$

The Taylor expansion of P

- Note that P is infinitely differentiable; as such it admits a Taylor expansion.
- So, we have the following second-Taylor-polynomial approximation for P around any i_0:

$$P(i) \approx P(i_0) + P'(i_0)(i - i_0) + \frac{1}{2} P''(i_0)(i - i_0)^2$$

  for i close to i_0
- The expressions for the derivatives are

$$P'(i) = -\sum_{t \geq 0} C_t \, t \, (1 + i)^{-1-t}$$

$$P''(i) = \sum_{t \geq 0} C_t \, t \, (t + 1) \, (1 + i)^{-2-t}$$

The Taylor expansion of P

- Note that P is infinitely differentiable; as such it admits a Taylor expansion.
- So, we have the following second-Taylor-polynomial approximation for P around any i_0:

$$P(i) \approx P(i_0) + P'(i_0)(i - i_0) + \frac{1}{2} P''(i_0)(i - i_0)^2$$

for i close to i_0

- The expressions for the derivatives are

$$P'(i) = -\sum_{t \geq 0} C_t \, t \, (1+i)^{-1-t}$$

$$P''(i) = \sum_{t \geq 0} C_t \, t \, (t+1) \, (1+i)^{-2-t}$$

The tangent (first-order) approximation of P

- If we are satisfied with a coarser approximation, we can neglect the second-order term in the Taylor polynomial and write

$$P(i) \approx P(i_0) + P'(i_0)(i - i_0)$$

- Then,

$$\frac{P(i) - P(i_0)}{P(i_0)} \approx \frac{P'(i_0)}{P(i_0)}(i - i_0)$$

for i close to i_0

- The expression above represents the fractional price change, i.e., the relative price change

The tangent (first-order) approximation of P

- If we are satisfied with a coarser approximation, we can neglect the second-order term in the Taylor polynomial and write

$$P(i) \approx P(i_0) + P'(i_0)(i - i_0)$$

- Then,

$$\frac{P(i) - P(i_0)}{P(i_0)} \approx \frac{P'(i_0)}{P(i_0)}(i - i_0)$$

for i close to i_0

- The expression above represents the fractional price change, i.e., the relative price change

The tangent (first-order) approximation of P

- If we are satisfied with a coarser approximation, we can neglect the second-order term in the Taylor polynomial and write

$$P(i) \approx P(i_0) + P'(i_0)(i - i_0)$$

- Then,

$$\frac{P(i) - P(i_0)}{P(i_0)} \approx \frac{P'(i_0)}{P(i_0)}(i - i_0)$$

  for i close to i_0

- The expression above represents the fractional price change, i.e., the relative price change

Modified duration

- For any $i > -1$, we define the modified duration (of the price P) by

$$D(i,1) = -\frac{P'(i)}{P(i)}$$

- Then,

$$\frac{P(i) - P(i_0)}{P(i_0)} \approx -D(i_0,1)(i - i_0)$$

  for i close to i_0
- Explicitly,

$$D(i,1) = \frac{\sum_{t \geq 0} C_t \, t \, (1+i)^{-t-1}}{\sum_{t \geq 0} C_t \, (1+i)^{-t}}$$

- Let us look at the modified duration in several familiar cases ...

Modified duration

- For any $i > -1$, we define the modified duration (of the price P) by

$$D(i,1) = -\frac{P'(i)}{P(i)}$$

- Then,

$$\frac{P(i) - P(i_0)}{P(i_0)} \approx -D(i_0,1)(i - i_0)$$

for i close to i_0

- Explicitly,

$$D(i,1) = \frac{\sum_{t\geq 0} C_t\, t\,(1+i)^{-t-1}}{\sum_{t\geq 0} C_t\,(1+i)^{-t}}$$

- Let us look at the modified duration in several familiar cases . . .

Modified duration

- For any $i > -1$, we define the modified duration (of the price P) by

$$D(i,1) = -\frac{P'(i)}{P(i)}$$

- Then,

$$\frac{P(i) - P(i_0)}{P(i_0)} \approx -D(i_0,1)(i - i_0)$$

  for i close to i_0

- Explicitly,

$$D(i,1) = \frac{\sum_{t \geq 0} C_t \, t \, (1+i)^{-t-1}}{\sum_{t \geq 0} C_t \, (1+i)^{-t}}$$

- Let us look at the modified duration in several familiar cases . . .

Modified duration

- For any $i > -1$, we define the modified duration (of the price P) by

$$D(i,1) = -\frac{P'(i)}{P(i)}$$

- Then,

$$\frac{P(i) - P(i_0)}{P(i_0)} \approx -D(i_0,1)(i - i_0)$$

  for i close to i_0
- Explicitly,

$$D(i,1) = \frac{\sum_{t\geq 0} C_t \, t \, (1+i)^{-t-1}}{\sum_{t\geq 0} C_t \, (1+i)^{-t}}$$

- Let us look at the modified duration in several familiar cases . . .

Modified duration with $i^{(m)}$ and δ

- For any $i > -1$, we define the modified duration $D(i, m)$ (of the price P) by

$$D(i,m) = -\frac{\frac{d}{d\,i^{(m)}}P(i)}{P(i)}$$

where $i^{(m)} = m[(1+i)^{1/m} - 1]$, i.e., $i^{(m)}$ is the nominal interest rate convertible $m-$thly which is equivalent to the annual effective interest rate i

- Then,

$$D(i,m) = \frac{1+i}{1+\frac{i^{(m)}}{m}} \cdot D(i,1)$$

- Rearranging the terms above yields the definition of the Macaulay duration

$$D(i,\infty) = D(i,1)(1+i) = \sum_{t \geq 0}\left[\left(\frac{C_t\,(1+i)^{-t}}{P(i)}\right)t\right]$$

- So, it's yet another weighted average ...

Modified duration with $i^{(m)}$ and δ

- For any $i > -1$, we define the modified duration $D(i,m)$ (of the price P) by

$$D(i,m) = -\frac{\frac{d}{d\,i^{(m)}}P(i)}{P(i)}$$

  where $i^{(m)} = m[(1+i)^{1/m} - 1]$, i.e., $i^{(m)}$ is the nominal interest rate convertible $m-$thly which is equivalent to the annual effective interest rate i

- Then,

$$D(i,m) = \frac{1+i}{1+\frac{i^{(m)}}{m}} \cdot D(i,1)$$

- Rearranging the terms above yields the definition of the Macaulay duration

$$D(i,\infty) = D(i,1)(1+i) = \sum_{t\geq 0}\left[\left(\frac{C_t\,(1+i)^{-t}}{P(i)}\right)t\right]$$

- So, it's yet another weighted average ...

Modified duration with $i^{(m)}$ and δ

- For any $i > -1$, we define the modified duration $D(i, m)$ (of the price P) by

$$D(i,m) = -\frac{\frac{d}{d\,i^{(m)}}P(i)}{P(i)}$$

where $i^{(m)} = m[(1+i)^{1/m} - 1]$, i.e., $i^{(m)}$ is the nominal interest rate convertible $m-$thly which is equivalent to the annual effective interest rate i

- Then,

$$D(i,m) = \frac{1+i}{1+\frac{i^{(m)}}{m}} \cdot D(i,1)$$

- Rearranging the terms above yields the definition of the Macaulay duration

$$D(i,\infty) = D(i,1)(1+i) = \sum_{t\geq 0}\left[\left(\frac{C_t\,(1+i)^{-t}}{P(i)}\right) t\right]$$

- So, it's yet another weighted average ...

Modified duration with $i^{(m)}$ and δ

- For any $i > -1$, we define the modified duration $D(i,m)$ (of the price P) by

$$D(i,m) = -\frac{\frac{d}{d\,i^{(m)}} P(i)}{P(i)}$$

  where $i^{(m)} = m[(1+i)^{1/m} - 1]$, i.e., $i^{(m)}$ is the nominal interest rate convertible $m-$thly which is equivalent to the annual effective interest rate i
- Then,

$$D(i,m) = \frac{1+i}{1+\frac{i^{(m)}}{m}} \cdot D(i,1)$$

- Rearranging the terms above yields the definition of the Macaulay duration

$$D(i,\infty) = D(i,1)(1+i) = \sum_{t \geq 0} \left[\left(\frac{C_t\,(1+i)^{-t}}{P(i)} \right) t \right]$$

- So, it's yet another weighted average ...