Even More on The Growth of Money

1. Nominal Rates of Interest and Discount

2. Constant Force of Interest

3. Force of Interest

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What is the nominal (annual) interest rate?

Assume that the bank credits the interest more than once per year - say, m times in a single year.

- $i^{(m)}$ . . . nominal (annual) interest rate compounded (convertible, payable) m times per year
- The word "nominal" means that the interest rate $i^{(m)}$ is annual *in name only*, i.e., the mechanism is such that the bank pays interest at the rate of

$$\frac{i^{(m)}}{m} \text{ after each } m^{th} \text{ of a year}$$

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Connection between $i^{(m)}$ and i

- Let i be the (usual) effective interest rate for the above investment. It is also known as the annual percentage yield(APY). Then,

$$\frac{i^{(m)}}{m} = (1+i)^{1/m} - 1$$

- The above statement is equivalent to

$$i^{(m)} = m[(1+i)^{1/m} - 1]$$

and

$$i = \left(1 + \frac{i^{(m)}}{m}\right)^m - 1$$

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Connection between $i^{(m)}$ and i (cont'd)

- In words, for every invested dollar, one gets

$$1 + i$$

at the end of the year.

On the other hand, after every m^{th} of the year the money currently on the account grows by a factor of $(1 + i^{(m)}/m)$; there are m such compoundings in a single year, so the final amount of money equals

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Examples

I. Find the accumulated value of $500 invested for five years at 0.08 per annum convertible quarterly.

$\Rightarrow$

$$500[1 + \frac{0.08}{4}]^{4 \cdot 5} = 500 \cdot 1.02^{20} = 742.97$$

- *Note:* The above investment scheme is equivalent to the one in which one invests $500 at 2% for 20 years.

II. How about an example involving **unknown interest rate**?

III. *Note:* The whole set-up works even for non-integer m

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The discount rate

- $d^{(m)}$ … nominal discount rate compounded (convertible, payable) m times per year
- With d as the annual discount rate, similarly as above, we have

$$1 - d = \left(1 - \frac{d^{(m)}}{m}\right)^m$$

or, equivalently.

$$d = 1 - \left(1 - \frac{d^{(m)}}{m}\right)^m$$

or

$$d^{(m)} = m\left[1 - (1-d)^{1/m}\right]$$

- For **equivalent** discount and interest rates, we have

$$\left(1 - \frac{d^{(m)}}{m}\right)\left(1 + \frac{i^{(m)}}{m}\right) = 1$$

- There are many different forms of this equality that are useful depending on the context; see equations 1.10.11 through 1.10.13 in the textbook and the appendix

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A basic equivalence equality

- The following is the equality which connects the nominal and effective discount and interest rates. This one is worth memorizing!

$$\left(1+\frac{i^{(n)}}{n}\right)^{n}=1+i=(1-d)^{-1}=\left(1-\frac{d^{(p)}}{p}\right)^{-p}$$

  for all positive integers n and p

- The above equality allows us to compare between the outcomes of different contracts.
- *Note:* If $i>0$ and $m>1$, then

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Force of Interest

- $\delta \ldots$ the force of interest, i.e.,

$$\delta := \lim_{m\to\infty} i^{(m)} = \ln(1+i)$$

- One should imagine that the compounding occurs at infinitesimally short time intervals ...
- With the above notation, the amount function takes the form

$$a(t) = e^{\delta t}$$

- *Note:* A mild manipulation of the terms in the limit yields

$$\delta = \lim_{m\to\infty} d^{(m)}$$

and we can append the sequence of inequalities from the end of the last section as follows:

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Examples

- Find the accumulated value of $1000 invested for 10 years if the force of interest equals 5%.

$\Rightarrow$

$$1000e^{0.05\cdot 10} = 1000e^{0.5} = 1648.72$$

- Now, an example that illustrates a comparison between the different rates we discussed so far ..
- *Assignment:* Problems 1.11.1-3

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- δ_t ... force of interest at time t; it is a **function of time** defined as

$$\delta_t = \frac{a'(t)}{a(t)} = \frac{d}{dt}[\ln(a(t))]$$

- *Special cases:*

I. **Compound interest.** If we are in the "ordinary" compound case, then the force of interest does not depend on the time variable t, i.e., it is constant and we are back in the realm of the previous section. Formally, we have

$$\delta_t = \frac{(1+i)^t \ln(1+i)}{(1+i)^t} = \ln(1+i) = \delta$$

II. **Simple interest.**

$$\delta_t = \frac{s}{1+st}$$

III. **Simple discount.**

$$\delta_t = \frac{-(1-d\,t)^{-2}\cdot(-d)}{(1-d\,t)^{-1}} = \frac{d}{1-d\,t}$$

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From δ_t to $a(t)$

- Recalling the definition of δ_t, and using the Fundamental Theorem of Calculus (simply put - we integrate both sides of the defining equality), we get

$$a(t) = e^{\int_0^t \delta_r\, dr} = \exp(\int_0^t \delta_r\, dr)$$

- For instance, let the time-varying force of interest be given as

$$\delta_t = \frac{1}{1+t}$$

Then the accumulated value of \$1 at the end of n years equals

$$a(n) = \exp(\int_0^n \delta_r\, dr) = \exp(\int_0^n \frac{1}{1+r}\, dr) = \exp\left(\ln(1+r)|_{r=0}^n\right) = 1+n.$$

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