# Brownian Motion and Ito's Lemma

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# The Black-Scholes Assumption About Stock Prices

- The original paper by Black and Scholes assumes that the price of the underlying asset is a stochastic process $\{S_t\}$ which is solves the following **stochastic differential equation** (in the differential form):

$$dS_t = S_t[\alpha \, dt + \sigma \, dZ_t]$$

  where

- $\alpha \ldots$ denotes the continuously compounded expected return on the stock;
- $\sigma \ldots$ denotes the volatility;
- $\{Z_t\} \ldots$ is a standard Brownian motion
- In other words, $\{S_t\}$ is a geometric Brownian motion

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# On the distribution of the stock price at a given time

- Recall the example from class to conclude that

$$\ln(S_t) \sim N\left(\ln(S_0) + (\alpha - \frac{1}{2})\sigma^2)t, \sigma^2 t\right), \text{ for every } t$$

- In other words, at any time $t$ the stock-price random variable $S_t$ is **log-normal**
- The above means that we assume that the continuously compounded returns are modeled by a normally distributed random variable.

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# Brownian Motion and Ito's Lemma

# More Heuristics: Relative Importance of the Drift and Noise

- Recall the SDE which defines the geometric B.M.

$$dS_t = S_t[\alpha\, dt + \sigma\, dZ_t]$$

- Consider a time period of length $h$ and the ratio of the per-period standard deviation to the per-period drift, i.e.,

$$\frac{\sigma S_t \sqrt{h}}{\alpha S_t h} = \frac{\sigma}{\alpha \sqrt{h}}$$

- For $h$ infinitesimaly small the above ration diverges.
- We may interpret this by saying that for short time-periods the "random component" of the process $\{S_t\}$ is dominant.
- As the observed period grows longer, the drift (mean) of the stochastic process $\{S_t\}$ has a greater effect

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# Brownian Motion and Ito's Lemma

# Theorem [Ito's Product Rule]

- Consider two Ito proocesses $\{X_t\}$ and $\{Y_t\}$. Then

$$d(X_t \cdot Y_t) = X_t\, dY_t + Y_t\, dX_t + dX_t\, dY_t.$$

- Note: We calculate the last term using the multiplication table with "$dt$'s" and "$dB_t$'s"

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# Brownian Motion and Ito's Lemma

# Martingality

- Under some integrability and regularity conditions on the integrand, the process $\{Y_t\}$ defined by

$$Y_t = \int_0^t \nu_s \, dB_S,$$

where $\{B_s\}$ is a standard B.M. is a martingale.

- In particular

$$\mathbb{E}[Y_t] = \mathbb{E}\left[\int_0^t \nu_s \, dB_S\right] = 0, \text{ for every } t$$

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# Ito Isometry

- Under some integrability and regularity conditions on the integrand $\nu$, let us define the process $\{Y_t\}$ as

$$Y_t = \int_0^t \nu_s \, dB_S,$$

where $\{B_s\}$ is a standard B.M. Then

$$\mathbb{E}[Y_t^2] = \mathbb{E}[\int_0^t \nu_s^2 \, ds]$$

# Continuity

- Under some integrability and regularity conditions on the integrand $\nu$, let us define the process $\{Y_t\}$ as

$$Y_t = \int_0^t \nu_s \, dB_S,$$

where $\{B_s\}$ is a standard B.M. Then the paths of $\{Y_t\}$ are (almost surely) continuous.

# Linearity

- Moreover, for a constant $c$, we have that

$$cY_t = \int_0^t (c\nu_s)\, dB_S,$$

- Additionally, if $\{A_t\}$ is a stochastic process given as

$$A_t = \int_0^t \xi_s\, dB_s,$$

for an integrand $\{\xi_t\}$ conforming to the integrability and regularity conditions necessary for the sotchastic integral to be well-defined, then

$$Y_t \pm A_t = \int_0^t (\nu_s \pm \xi_S)\, dB_S$$

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## The Set-Up

- Consider two stock prices $\{S_t\}$ and $\{Q_t\}$. Suppose that they satisfy the following system of SDEs

$$\frac{dS_t}{S_t} = \alpha_S \, dt + \sigma_s \, dW_t$$
$$\frac{dQ_t}{Q_t} = \alpha_Q \, dt + \sigma_Q \left[\rho dW_t + \sqrt{1-\rho^2} \, dW'_t\right]$$

  where $\rho \in [-1, 1], \alpha_S, \alpha_Q, \sigma_S > 0$ and $\sigma_Q > 0$ are given constants and $\{W_t\}$ and $\{W'_t\}$ are **independent** standard Brownian motions.
- **Theorem:** If $W$ and $W'$ are independent, then $dW_t \, dW'_t = 0$.
- We can now add the above to our multiplication table.

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- **Theorem:** If $W$ and $W'$ are independent, then $dW_t\, dW_t' = 0$.
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# A New Standard Brownian Motion

- Define

$$\tilde{W}_t = \rho W_t + \sqrt{1-\rho^2}\, W'_t.$$

- $\{\tilde{W}_t\}$ is an almost everywhere continuous process with $\tilde{W}_0 = 0$
- One can prove that $\tilde{W}$ is a standard Brownian motion. Now, we can write

$$\frac{dQ_t}{Q_t} = \alpha_Q\, dt + \sigma_Q\, d\tilde{W}_t$$

- According to Ito's Product Rule and the fact that $W$ and $W'$ are independent

$$d(W_t \tilde{W}_t) = W_t\, d\tilde{W}_t + \tilde{W}_t\, dW_t + \rho\, dt$$

- In the integral form the above reads as

$$W_t \tilde{W}_t = \int_0^t W_s\, d\tilde{W}_s + \int_0^t \tilde{W}_s\, dW_s + \rho t$$

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# On the Correlation of the two Brownian Motions

- For covenience, let us repeat that

$$W_t \tilde{W}_t = \int_0^t W_s \, d\tilde{W}_s + \int_0^t \tilde{W}_s \, dW_s + \rho t$$

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# The Ornstein-Uhlenbeck Process

- Along with the processes we discussed so far, consider a stochastic process $\{X_t\}$ which satisfies

$$dX_t = [\alpha - x_t]\, dt + \sigma\, dZ_t,$$

  where $\alpha$ and $\sigma$ are given constants and $\{Z_t\}$ is a standard Brownian motion.
- The process above is called the **mean reverting process** (Why??)
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